P122 | Advanced Topics in Plasma Physics

Drift Kinetics and Quasilinear Theory

From kinetic response to the evolution of the distribution

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CGL follows density and two pressures, but particles with different parallel velocities can respond very differently to the same perturbation. Drift kinetics retains this velocity dependence while averaging over rapid gyromotion. We first identify the guiding-center motions that it describes and explain how the drift-kinetic equation emerges from the ordering of a strongly magnetized plasma.

We then use a simple electrostatic kinetic model to study how waves respond to a given distribution, how phase mixing and Landau resonance transfer energy in velocity space, and how field–particle correlations can diagnose that transfer. Finally, quasilinear theory describes the cumulative effect of many weak resonant interactions as diffusion in velocity space. The bump-on-tail problem provides a concrete example in which growing waves reshape the distribution that drives them and produce a plateau near marginal stability.

The organizing questions are: what determines a wave's growth or damping, where does a damped perturbation go in phase space, and how does the wave change the background distribution until its own drive is removed?

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1 From CGL to drift kinetics

1.1 Why keep the distribution function?

The CGL laws,

$$\frac{D}{Dt}\left(\frac{p_\perp}{nB}\right)=0, \qquad \frac{D}{Dt}\left(\frac{p_\parallel B^2}{n^3}\right)=0, \tag{1.1}$$

neglect the relevant heat-flux terms. Collisionless particles can nevertheless carry heat along the field: in a hot region connected to cooler plasma, fast particles leave sooner than slow ones. If $L_\parallel/v_{\rm th}$ is comparable to the evolution time, this streaming matters. A few moments also do not specify the slope of the distribution near a wave's resonant velocity. Both problems require more information about f.

1.2 Average over gyromotion, retain guiding-center motion

A systematic asymptotic reduction starts from the *magnetized Vlasov equation*, coupled to Maxwell's equations (or Poisson's equation for electrostatic perturbations with a prescribed magnetic field). With consistently ordered weak fluctuations, low frequencies, and long parallel scales, one standard route is

$$\text{Vlasov (6D)} \xrightarrow[\text{average gyrophase}]{|\omega|\ll\Omega_c} \underset{k_\perp\rho\sim1}{\text{gyrokinetics (5D)}} \xrightarrow{k_\perp\rho\ll1} \text{drift kinetics (5D)}.$$

Here Ω_c is the gyrofrequency and ρ the thermal gyroradius; the parallel ordering is $|k_\parallel|v_{\rm th} \ll \Omega_c$. Eliminating the rapid gyroangle leaves three orbit-center position coordinates and two velocity coordinates, $v_\parallel$ and μ. The long-wavelength limit removes finite-gyroradius averaging effects at leading order.

We use a heuristic derivation below: take the leading-order guiding-center motions, then obtain the kinetic equation by conserving the collisionless distribution along those trajectories.

We consider the **simplest electrostatic slab form of drift kinetics**: a straight, spatially uniform, time-independent magnetic field, $\boldsymbol{B}=B_0\widehat{\boldsymbol{z}}$, with $\delta\boldsymbol{B}=0$ and weak electrostatic perturbations. There are no magnetic-field gradients or curvature, so mirror forces and magnetic drifts vanish. Spatial gradients of f and $\boldsymbol{E}$ are still allowed at this stage.

For a magnetized species, take

$$\frac{|\omega|}{\Omega_c}\ll1, \qquad k_\perp\rho\ll1, \qquad \frac{|k_\parallel|v_{\rm th}}{\Omega_c}\ll1, \qquad \Omega_c=\frac{|q|B}{mc}, \quad \rho=\frac{v_{{\rm th},\perp}}{\Omega_c}. \tag{1.2}$$

The rapid gyrophase can then be averaged out. Use

$$f=f(\boldsymbol{R},v_\parallel,\mu,t), \qquad \mu=\frac{mv_\perp^2}{2B}, \qquad \mathrm{d}^3v=\frac{2\pi B}{m}\,\mathrm{d}\mu\,\mathrm{d}v_\parallel. \tag{1.3}$$

Here $\boldsymbol{R}$ is the guiding-center position and μ is conserved to the retained order. Removing gyrophase does not remove perpendicular energy: it is still represented by μB.

For this geometry, with $\boldsymbol{E}=-\boldsymbol{\nabla}\phi$, the leading-order guiding-center motions are

$$\dot{\boldsymbol{R}}=v_\parallel\widehat{\boldsymbol{z}}+\boldsymbol{u}_E, \qquad \boldsymbol{u}_E=\frac{c\boldsymbol{E}\times\boldsymbol{B}}{B^2}, \qquad \dot{v}_\parallel=\frac{qE_\parallel}{m}, \qquad \dot{\mu}=0. \tag{1.4}$$

Conservation of the collisionless distribution along these trajectories gives

$$\begin{aligned} 0=\frac{\mathrm{d}f}{\mathrm{d}t}&=\partial_t f+\dot{\boldsymbol{R}}\cdot\boldsymbol{\nabla}f+\dot{v}_\parallel\partial_{v_\parallel}f+\dot{\mu}\partial_\mu f,\\ \Longrightarrow\quad 0&=\boxed{\partial_t f+v_\parallel\partial_z f+\boldsymbol{u}_E\cdot\boldsymbol{\nabla}_\perp f+\frac{qE_\parallel}{m}\partial_{v_\parallel}f.} \end{aligned} \tag{1.5}$$

The three transport terms represent parallel streaming, perpendicular advection, and acceleration through parallel velocity space. This is the slab drift-kinetic model we need.

In a weak-flow ordering, $u_E \ll v_{\mathrm{th}}$, but perpendicular advection can compete with streaming when

$$\frac{u_E}{v_{\mathrm{th}}} \sim \frac{k_\parallel}{k_\perp} \ll 1 \quad \Longrightarrow \quad k_\perp u_E \sim k_\parallel v_{\mathrm{th}}. \tag{1.6}$$

The comparison is between rates, not velocities alone. This is the same scale-based reasoning used in anisotropic turbulence.

1.3 Connection to the magnetic mirror

In a slowly varying, stationary magnetic flux tube, the parallel acceleration also contains the mirror force:

$$\dot{v}_\parallel = \frac{qE_\parallel}{m} - \frac{\mu}{m}\nabla_\parallel B. \tag{1.7}$$

For $E_\parallel = 0$, increasing B raises μB at the expense of parallel kinetic energy. General geometry also introduces magnetic drifts. We will keep $B = B_0$ below, so these geometric terms vanish and the parallel electric interaction can be followed on its own.

2 Equilibrium, fluctuations, and the kinetic model

2.1 One model for the rest of the calculation

We now make an **additional one-dimensional approximation**: set perpendicular spatial gradients to zero and integrate over perpendicular velocities. This removes the $\boldsymbol{u}_E \cdot \boldsymbol{\nabla}_\perp f$ term; the remaining parallel acceleration is due only to $E_\parallel$, so B_0 no longer appears explicitly. From here onward, $f(z, v, t)$ denotes this *one-dimensional* distribution, $v = v_\parallel$, and $E = E_\parallel$. Use an electron plasma with a uniform, immobile ion background of density n_0:

$$\boxed{\partial_t f + v\partial_z f + \frac{q}{m}E\partial_v f = 0, \qquad \partial_z E = 4\pi q\left(\int f\,\mathrm{d}v - n_0\right).} \tag{2.1}$$

The first equation follows particles; the second makes the field respond to their density. Take a periodic spatial domain, zero mean E, and distributions that vanish at large $|v|$.

The parallel kinetic equation has this form beyond the low-frequency drift ordering as well. For the rest of the main calculation we use the classical *Vlasov–Poisson* model, including Langmuir waves. Their frequency is not assumed to satisfy (1.2). The particle–field feedback developed below is the quasilinear method; it can later be applied to low-frequency modes using their own field equations.

2.2 Separate the mean from the fluctuation

Write

$$f = F(v, t) + \delta f(z, v, t), \qquad F = \langle f \rangle, \qquad \langle \delta f \rangle = 0. \tag{2.2}$$

Here $\langle g \rangle = L^{-1}\int_0^L g\,\mathrm{d}z$ averages over the periodic domain at fixed v. Periodicity removes the streaming term:

$$\langle v\partial_z f \rangle = \frac{v}{L}\left[f(L, v, t) - f(0, v, t)\right] = 0.$$

Free streaming redistributes particles in space without changing F. Since E is independent of v and $\langle E \rangle = 0$, the electric term becomes

$$\langle E\partial_v f \rangle = \langle E \rangle\, \partial_v F + \partial_v \langle E\delta f \rangle = \partial_v \langle E\delta f \rangle.$$

Thus averaging (2.1) gives the exact equation

$$\boxed{\partial_t F = -\frac{q}{m}\partial_v \langle E\delta f\rangle .} \tag{2.3}$$

For slow QLT evolution we will also average over rapid wave oscillations, with the window specified in Section 5.3.

Subtract (2.3) from the full kinetic equation:

$$(\partial_t + v\partial_z)\delta f = -\frac{q}{m}E\partial_v F - \frac{q}{m}\left(E\partial_v\delta f - \langle E\partial_v\delta f\rangle\right). \tag{2.4}$$

Streaming changes the spatial pattern carried by each velocity group. The first term on the right is the response to the background gradient; the second couples fluctuating fields and distributions.

Linear theory: hold F fixed and omit both second-order terms inside the parentheses in (2.4), retaining $-(q/m)E\partial_v F$ on the right.
Quasilinear theory: use that linear response about a slowly evolving F, but retain $\langle E\delta f\rangle$ in (2.3). The fluctuations can then change the distribution that determines their growth.

The distinction is a time-scale ordering: QLT requires mean feedback to act before neglected nonlinear fluctuation dynamics becomes important. We make this ordering explicit in Section 5.

3 Linear response: resonance, damping, and growth

3.1 Solve for the response and close the field equation

First freeze $F(v)$. For a Fourier component proportional to $e^{i(kz-\omega t)}$, with $k > 0$ and $\omega = \omega_r + i\gamma$, equation (2.4) becomes

$$-i(\omega - kv)\delta f_k = -\frac{q}{m}E_k F'(v) \quad\Longrightarrow\quad \boxed{\delta f_k = -i\frac{q}{m}\frac{E_k F'(v)}{\omega - kv}.} \tag{3.1}$$

The numerator says which part of the distribution responds; the denominator identifies particles that remain in step with the wave. Along $z = z_0 + vt$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(kz - \omega_r t) = kv - \omega_r \quad\Longrightarrow\quad \boxed{v_{\mathrm{res}} = \frac{\omega_r}{k}.} \tag{3.2}$$

Away from resonance the force alternates rapidly along the orbit. Near resonance, coherent work can accumulate over many wave periods.

Insert (3.1) into $ikE_k = 4\pi q\int \delta f_k\,\mathrm{d}v$. A nonzero wave requires

$$\boxed{\epsilon(\omega,k) = 1 - \frac{\omega_p^2}{n_0 k^2}\int_{\mathcal{L}}\frac{F'(v)}{v-\omega/k}\,\mathrm{d}v = 0, \qquad \omega_p^2 = \frac{4\pi n_0 q^2}{m}.} \tag{3.3}$$

The contour $\mathcal{L}$ specifies the causal initial-value response. The rule we need, for real ω_r, is

$$\int_{\mathcal{L}}\frac{F'(v)}{v-\omega_r/k}\,\mathrm{d}v = \mathcal{P}\int\frac{F'(v)}{v-\omega_r/k}\,\mathrm{d}v + i\pi F'(\omega_r/k). \tag{3.4}$$

$\mathcal{P}$ denotes the principal value. The principal-value part sets the oscillation frequency; the resonant term permits damping or growth. This rule follows by starting with $\mathrm{Im}\,\omega > 0$ and continuing the response while keeping the contour below the pole. We use the rule without performing the full contour calculation.

3.2 The sign of the resonant slope

For $|\gamma| \ll \omega_r$, separate real and imaginary parts:

$$\epsilon_r(\omega_r, k) = 0, \qquad \gamma = -\frac{\epsilon_i(\omega_r, k)}{\partial \epsilon_r / \partial \omega_r}. \tag{3.5}$$

For a Maxwellian electron core and a small tail population, take the positive-frequency, long-wavelength Langmuir branch. To leading order,

$$\epsilon_r \simeq 1 - \frac{\omega_p^2}{\omega_r^2}, \qquad \omega_r \simeq \omega_p, \qquad \frac{\partial \epsilon_r}{\partial \omega_r} \simeq \frac{2}{\omega_p}. \tag{3.6}$$

Using (3.4) in (3.5) gives

$$\boxed{\gamma_k \simeq \frac{\pi \omega_p^3}{2 n_0 k^2} F'\left(\frac{\omega_p}{k}\right).} \tag{3.7}$$

A decreasing tail damps this wave; a positive-slope region can drive it. The beam must be weak enough that the core still sets approximately the Langmuir frequency. Thermal corrections to ω_r are needed for an accurate Maxwellian damping rate; the leading expression here establishes the resonant slope dependence.

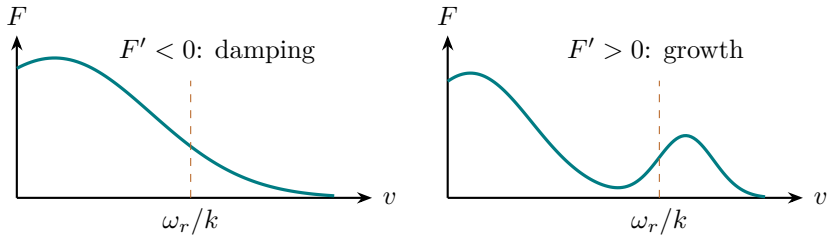


Figure 1: The same resonant response gives damping or instability, depending on the distribution. Only the positive-velocity side is shown.

This is an ensemble energy-exchange statement: the imbalance of resonant particles on either side of the phase speed determines the net work. An individual particle can gain or lose energy depending on its phase in the wave.

> Linear theory predicts what waves do to a *given* distribution. It leaves two questions: where does the perturbation go when the wave damps, and what eventually stops growth when the distribution is unstable?

4 Energy exchange, phase mixing, and irreversibility

4.1 Where does the electric-field energy go?

Define the spatially averaged particle and electric energies per unit volume,

$$K = \left\langle \int \frac{m v^2}{2} f \, \mathrm{d}v \right\rangle, \qquad U_E = \frac{\langle E^2 \rangle}{8\pi}, \qquad J = q \int v f \, \mathrm{d}v. \tag{4.1}$$

Multiply the kinetic equation by $mv^2/2$ and integrate. Streaming integrates to zero in the periodic domain, while integration by parts in v gives

$$\frac{\mathrm{d}K}{\mathrm{d}t} = -q \left\langle E \int \frac{v^2}{2} \partial_v f \, \mathrm{d}v \right\rangle = \langle JE \rangle, \qquad \frac{\mathrm{d}U_E}{\mathrm{d}t} = -\langle JE \rangle. \tag{4.2}$$

For the field-energy identity, use charge continuity, $q\partial_t n + \partial_z J = 0$, obtained by integrating Vlasov over velocity, with $n = \int f \, \mathrm{d}v$. Differentiating Poisson's equation in time gives

$$\partial_z \partial_t E = 4\pi q \partial_t n = -4\pi \partial_z J \quad \Longrightarrow \quad \partial_t E = -4\pi \big(J - \langle J \rangle\big),$$

where the spatially constant term is fixed by $\langle E \rangle = 0$. Multiply by $E/(4\pi)$ and average:

$$\frac{\mathrm{d}U_E}{\mathrm{d}t} = \frac{1}{4\pi} \langle E \partial_t E \rangle = -\langle JE \rangle + \langle J \rangle \langle E \rangle = -\langle JE \rangle .$$

This is the electrostatic energy balance corresponding to Poynting's theorem.

Thus $K + U_E$ is conserved. Damping transfers field energy to particles, but this does not yet establish irreversible thermalization.

4.2 The distribution retains a ballistic part

Free streaming simply translates the initial pattern: $\delta f(z, v, t) = \delta f(z - vt, v, 0)$. An initial Fourier component $A(v)e^{ikz}$ therefore becomes $A(v)e^{ik(z-vt)}$, giving the factor e^{-ikvt} in δf_k.

Including the electric-field drive, the linear kinetic equation has the time-domain solution, obtained with the integrating factor e^{ikvt}:

$$\delta f_k(v, t) = \delta f_k(v, 0)e^{-ikvt} - \frac{q}{m} F'(v) \int_0^t E_k(t')e^{-ikv(t-t')} \, \mathrm{d}t'. \tag{4.3}$$

Even when the macroscopic field decays, the solution generally retains a ballistic contribution proportional to e^{-ikvt}. It oscillates increasingly rapidly in v:

$$\Delta v \sim \frac{1}{|k|t}. \tag{4.4}$$

Poisson's equation depends on $\int \delta f_k \, \mathrm{d}v$, so fine positive and negative contributions can cancel while δf_k itself remains finite.

To see this directly, switch off the field for a free-streaming illustration. Then

$$f(z, v, t) = f(z - vt, v, 0), \qquad \delta f = aF_M(v) \cos[k(z - vt)], \tag{4.5}$$

where $F_M = n_0 e^{-v^2/v_{\mathrm{th}}^2}/(\sqrt{\pi} v_{\mathrm{th}})$ and the initial perturbation is $aF_M \cos kz$. The Gaussian velocity integral gives

$$\delta n(z, t) = an_0 \cos(kz) e^{-(kv_{\mathrm{th}}t)^2/4}. \tag{4.6}$$

Every particle still has its original velocity. The density perturbation disappears through phase cancellation. Landau damping adds the self-consistent field and its energy exchange to this phase-mixing dynamics.

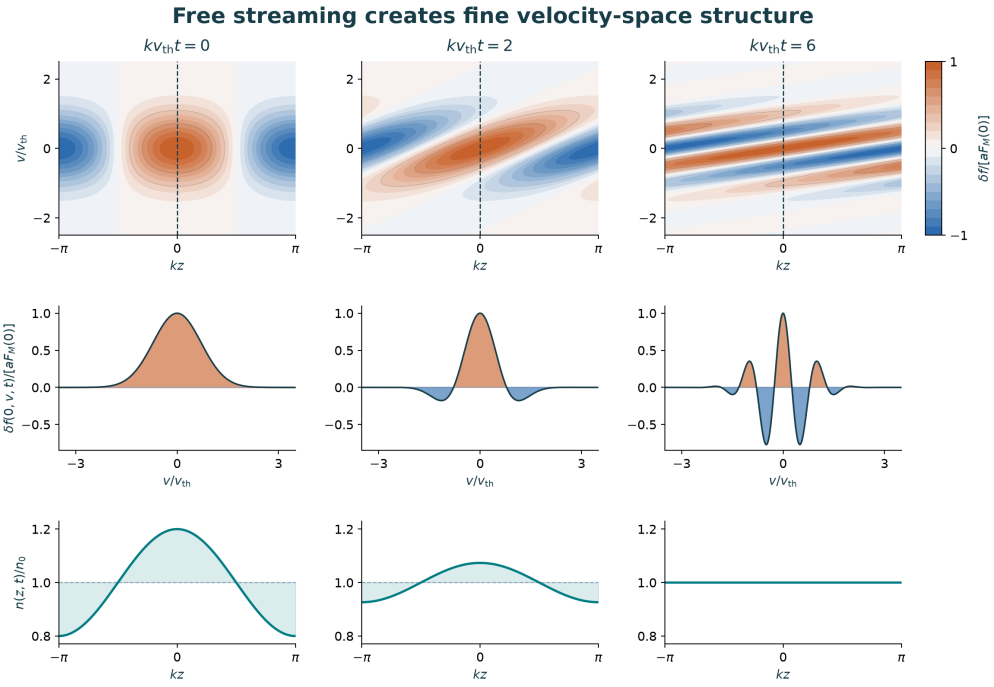


Figure 2: Free-streaming phase mixing for $\delta f = aF_M(v)\cos[k(z - vt)]$, with $F_M(v) = n_0 \exp(-v^2/v_{\rm th}^2)/(\sqrt{\pi} v_{\rm th})$ and $E = 0$. Columns show $kv_{\rm th}t = 0, 2, 6$. Top: contours of the phase-space perturbation. Middle: velocity cuts at $z = 0$, marked by the dashed lines above; orange and blue indicate positive and negative perturbations. Bottom: the spatial density profile, $n(z,t)/n_0 = 1 + a\exp[-(kv_{\rm th}t)^2/4]\cos(kz)$, for $a = 0.2$. Each row uses the same axis scales across all three times. Positive and negative contributions increasingly cancel in the velocity integral, so the density approaches its uniform mean while fine-scale structure remains in the distribution.

4.3 Free energy and the role of weak collisions

For small perturbations about a homogeneous Maxwellian, the linear collisionless system conserves the positive perturbation free energy

$$\boxed{\mathcal{W} = \frac{\langle E^2 \rangle}{8\pi} + \frac{T}{2}\left\langle \int \frac{(\delta f)^2}{F_M}\,\mathrm{d}v \right\rangle, \qquad \frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} = 0.} \tag{4.7}$$

As field energy decreases, free energy accumulates in the distribution. Unlike $\int \delta f\,\mathrm{d}v$, the integral of $(\delta f)^2/F_M$ does not cancel when the distribution develops fine structure. This is the kinetic extension of an energy budget; it also explains what low-order fluid moments miss.

Fine-grained Vlasov entropy is conserved. Collisions can remove the fine structure and produce entropy. Estimate this smoothing as velocity-space diffusion with coefficient $\sim \nu v_{\rm th}^2$. For the ballistic part $\delta f_k = A(v)e^{-ikvt}$,

$$\partial_v^2 e^{-ikvt} = -(kt)^2 e^{-ikvt}.$$

If A varies on scale $v_{\rm th}$, derivatives of this phase dominate when $|k|v_{\rm th}t \gg 1$, giving $\partial_v^2 \delta f_k \simeq -(kt)^2 \delta f_k$. Thus

$$C[\delta f] \sim \nu v_{\rm th}^2 \partial_v^2 \delta f \sim -\nu (kv_{\rm th}t)^2 \delta f, \tag{4.8}$$

so the collisional attenuation becomes important when

$$\int_0^{t_c} \nu (kv_{\rm th}t)^2\,\mathrm{d}t \sim 1 \quad \Longrightarrow \quad \boxed{t_c \sim [\nu (kv_{\rm th})^2]^{-1/3}.} \tag{4.9}$$

For $\nu \ll |k| v_{\mathrm{th}}$, this is much shorter than ν^{-1}. Small collision frequency need not mean negligible collisions once velocity gradients become large.

Plasma echo. Before collisions erase that structure, a second perturbation can rearrange the phases left by the first and produce a later macroscopic signal. This echo demonstrates that phase mixing has hidden information in velocity space. Nonlinear mixing and unmixing will reappear in kinetic turbulence; no echo calculation is needed for the QLT derivation below.

4.4 Example: measuring energy transfer with field–particle correlations

The energy equation above describes exchange between particles and the electric field, but the integrated work JE does not identify which particles participate. A field–particle correlation (FPC) retains this velocity-space information. In the one-dimensional model, let $w = mv^2 f/2$. Multiplying the kinetic equation by $mv^2/2$ gives

$$\partial_t w + v \partial_z w = -q \frac{v^2}{2} E \partial_v f. \tag{4.10}$$

At a measurement point, average the electric-force term over a time window T_{av}:

$$\boxed{C(v) = -\left\langle q \frac{v^2}{2} E \partial_v f \right\rangle_{T_{\mathrm{av}}}, \qquad \int C(v)\, \mathrm{d}v = \langle JE \rangle_{T_{\mathrm{av}}}.} \tag{4.11}$$

Positive and negative values of $C(v)$ show where particles at a given velocity gain and lose energy through the electric force. Choosing T_{av} longer than the wave period suppresses reversible oscillatory exchange and exposes a persistent resonant signature, while the velocity integral recovers the net work.

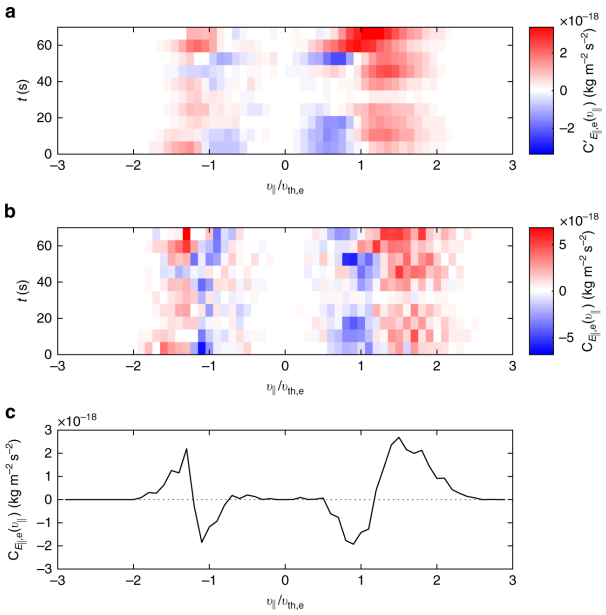


Figure 3: An example from MMS measurements in Earth's magnetosheath. The persistent positive and negative lobes of the electron FPC occur near the expected resonant velocities and are consistent with electron Landau damping. Reproduced from Ref. [7], Figure 2, under CC BY 4.0.

FPC is a diagnostic obtained directly from the kinetic energy equation; it does not require quasilinear theory. If the QLT assumptions hold, however, the measured velocity-space pattern can be compared with the diffusion prediction

$$\langle C \rangle_z \simeq \frac{mv^2}{2} \partial_t F = \frac{mv^2}{2} \partial_v (D \partial_v F). \tag{4.12}$$

Thus FPC tests where wave–particle energy transfer occurs, whereas QLT supplies a statistical closure for the slow evolution of F. A nonzero FPC establishes energy transfer, but by itself does not prove irreversible thermalization.

5 Quasilinear theory: coupled particles and waves

Linear theory holds F fixed and predicts whether a wave grows or damps. Quasilinear theory (QLT) lets the waves change F, and then uses the changed distribution to update their growth. This feedback can remove the instability's own drive.

Two possible outcomes near resonance. A particle responds to the total electric field, even though resonance is most easily described one Fourier component at a time. For mode k, define its phase $\psi_k = kz - \omega_k t$ and phase velocity $v_{\phi k} = \omega_k/k$. Its potential can produce an island centered near a stable wave phase and $v = v_{\phi k}$. In a multiwave field, islands associated with different resonances coexist. Figure 4 therefore uses one schematic phase coordinate and the absolute particle velocity v to show the whole picture at once. Suppose that a particle is initially close to resonance with a component k_0, so that $\omega_0 - k_0 v \simeq 0$. The field changes the particle velocity, and therefore changes both its mismatch with k_0 and the part of the spectrum that is now closest to resonance. On a Langmuir branch with $\omega \simeq \omega_p$, for example, the resonant wavenumber shifts according to $k_{\rm res} \simeq \omega_p/v$.

There are two limiting possibilities. **Coherent trapping:** if a wave is sufficiently strong and remains coherent long enough, the particle stays inside one island and bounces about its phase velocity. This requires a bounce time no longer than the field-correlation time, $\tau_b \lesssim \tau_c$; Section 7.1 gives the quantitative test. **Successive resonant kicks:** resonance overlap, additional modes, or time dependence can instead carry a particle across a separatrix and into the neighborhood of another resonance. If the force phase at the new encounter is no longer predictable from the preceding one, the successive changes act as random kicks. It is this loss of phase memory that is called decorrelation; the resonance islands themselves were already present. QLT describes the cumulative effect of this second possibility.

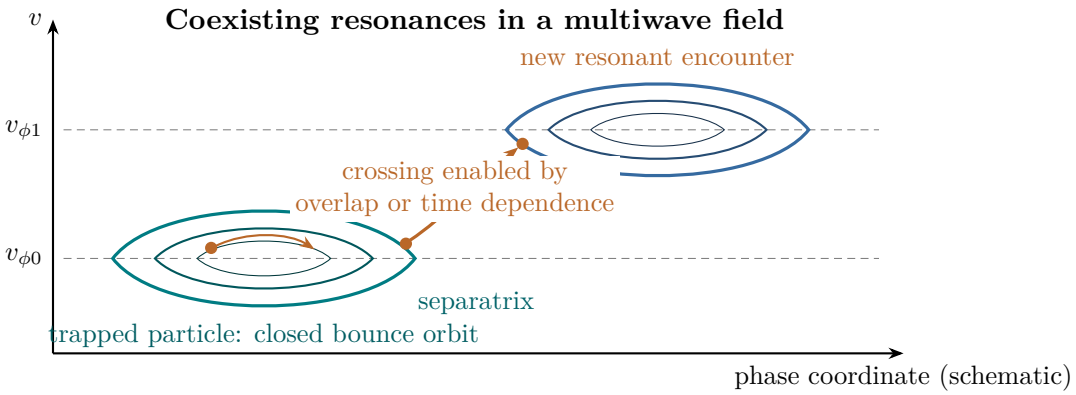


Figure 4: A single schematic phase-space portrait containing two resonance islands that coexist. One particle remains on a closed orbit inside the lower island and is coherently trapped. Another particle can cross a separatrix when additional modes, resonance overlap, or time dependence breaks the single-wave invariant, and can then encounter a different resonance. If the force phase at successive encounters is uncorrelated, their cumulative effect is velocity-space diffusion.

5.1 Resonant kicks: the particle picture before the calculation

Before deriving an equation for F, consider what an individual particle experiences. To lowest order, a particle with velocity v follows the unperturbed orbit

$$z(t) = z_0 + vt, \qquad \Delta v(t) = \frac{q}{m}\int_0^t E[z(s), s], \mathrm{d}s. \tag{5.1}$$

For one Fourier component, $E_k \propto \exp[i(kz - \omega t)]$, the phase sampled along this orbit is

$$kz(t) - \omega t = kz_0 + (kv - \omega)t. \tag{5.2}$$

Far from resonance, $|kv - \omega|$ is large: the particle rapidly samples electric fields of alternating sign, so most of the acceleration cancels. Near $kv = \omega$, the phase changes slowly and the electric force can act in nearly the same direction for a longer time. Resonant particles therefore receive the largest velocity changes.

We now follow the random-kick branch. Let τ_c be the decay time of the total electric-force correlation along the particle orbit. During one correlated encounter the particle receives a small kick, Δv_j. Later encounters have effectively unrelated phases, so the signs of successive kicks are random. A single perfectly coherent wave belongs instead to the trapping branch discussed above.

$$\Delta v = \Delta v_1 + \Delta v_2 + \cdots + \Delta v_N, \qquad N \sim \frac{t}{\tau_c}. \tag{5.3}$$

The mean kick can vanish while its variance grows:

$$\boxed{\langle \Delta v \rangle \simeq 0, \qquad \left\langle (\Delta v)^2 \right\rangle \simeq N \left\langle (\Delta v_j)^2 \right\rangle \equiv 2D(v)t.} \tag{5.4}$$

Thus diffusion describes the cumulative statistical effect of many weak resonant interactions. The coefficient $D(v)$ is largest for velocities that remain in phase with part of the wave spectrum.

The separation of time scales makes this a slow evolution of the background. Let δv_c be a typical kick received during one correlation time and let Δv_F be the velocity scale over which F changes appreciably. For weak kicks,

$$\delta v_c \ll \Delta v_F, \qquad D \sim \frac{(\delta v_c)^2}{\tau_c}, \qquad t_{\rm eq} \sim \frac{(\Delta v_F)^2}{D} \sim \tau_c \left(\frac{\Delta v_F}{\delta v_c} \right)^2 \gg \tau_c. \tag{5.5}$$

Thus F is effectively fixed during one encounter, even though many encounters can change it over $t_{\rm eq}$. The wave intensity must likewise change little during one encounter, so $\tau_c \ll \tau_{\rm g}$, where $\tau_{\rm g}$ is its growth time.

This is the regime addressed by QLT. On a fast time scale, solve for the fluctuation δf as a linear response to E about an almost fixed F. On the much longer time $t_{\rm eq}$, retain the small correlation of those fluctuations with the field. The exact averaged kinetic equation already has the form of a continuity equation in velocity space,

$$\boxed{\partial_t F + \partial_v \Gamma_v = 0, \qquad \Gamma_v \equiv \frac{q}{m} \langle E\, \delta f \rangle .} \tag{5.6}$$

Because the linear response has $\delta f = O(E)$, this mean flux is $O(E^2)$. The first-order oscillatory response is fast and averages away, while the second-order correlation can accumulate secularly and modify F.

Schematically, the averaging interval used by QLT lies between the fast wave and force-correlation times and the slow evolution times:

$$\boxed{\max(\tau_{\rm w}, \tau_c) \ll \Delta t \ll \min(t_{\rm eq}, \tau_{\rm g}), \qquad t_{\rm eq} \ll t_{\rm nl}.} \tag{5.7}$$

Here $t_{\rm nl}$ is the time on which the neglected nonlinear fluctuation dynamics changes the waves themselves. The final inequality says that the cumulative mean flux becomes important before that omitted dynamics destroys the linear-response description. Section 5.3 states these assumptions more precisely.

What direction does the flux take? Random kicks move particles both ways, but more particles leave a velocity interval on the side where F is larger. The diffusion picture therefore anticipates a flux down the gradient,

$$\Gamma_v = -D(v)\, \partial_v F. \tag{5.8}$$

Particle conservation in velocity space then gives

$$\boxed{\partial_t F + \partial_v \Gamma_v = 0 \quad \Longrightarrow \quad \partial_t F = \partial_v (D\,\partial_v F)\,.} \tag{5.9}$$

This is the simplest velocity-space Fokker–Planck equation. At this stage, only the down-gradient closure for the exact flux is a physical expectation. The calculation below will use the QLT linear response to obtain $\Gamma_v = -D\partial_v F$ and determine $D(v)$ from the wave spectrum.

> Resonance identifies which particles receive the largest kicks. Loss of phase coherence makes successive weak kicks statistically independent. Their mean can cancel while their variance grows linearly in time, producing diffusion rather than coherent acceleration.

5.2 The two coupled evolution equations

The unknowns are the mean distribution $F(v,t)$ and the electric-field intensity of each Fourier mode, $I_k(t) \equiv |E_k(t)|^2$. For the weak bump-on-tail problem, their evolution takes the form

> **The core of QLT: evolve the particles and the wave spectrum together.**
>
> $$\partial_t F = \partial_v (D\,\partial_v F)\,, \qquad \text{waves redistribute particles,} \tag{5.10a}$$
> $$\frac{\mathrm{d}I_k}{\mathrm{d}t} = 2\gamma_k[F] I_k, \qquad \text{particles drive or damp waves.} \tag{5.10b}$$
>
> The notation $\gamma_k[F]$ means that the growth rate is calculated from the current distribution $F(v,t)$. For the bump-on-tail problem, this dependence is given explicitly by (3.7). The wave spectrum sets D, and F sets the growth rates: each equation changes the input to the other.

This is a statistical reduction of the Vlasov–Poisson system (2.1). The linear dispersion relation (3.3), evaluated using the current F, supplies $\omega_{r,k}[F]$ and $\gamma_k[F]$. The field is represented by its spectrum $\{I_k\}$ rather than by following every wave phase.

Where is δf in this formulation? We solve for its linear response to E and F, then substitute that response into Poisson's equation and the mean feedback $\langle E\delta f\rangle$. This eliminates δf as an independent unknown in the reduced system, while retaining its effects in the wave response and D. In particular, $\langle \delta f\rangle = 0$ does not imply $\langle E\delta f\rangle = 0$: this correlation is what changes F.

Although the fluctuation response is linear in E_k, the coupled evolution is nonlinear: the waves change the distribution that determines their own growth. We now specify when this reduction is valid and derive its diffusion coefficient [1].

5.3 Which time scales must be separated?

The waves can change the *mean distribution*, or nonlinear interactions can change the *fluctuations themselves*. To distinguish their time scales, repeat the exact equations (2.3) and (2.4):

$$\partial_t F = \underbrace{-\frac{q}{m}\partial_v \langle E\delta f\rangle}_{\text{mean feedback: } t_{\mathrm{eq}} \text{ (retain)}}\,.$$

$$(\partial_t + v\partial_z)\delta f = -\frac{q}{m}E\partial_v F \underbrace{-\frac{q}{m}\left(E\partial_v\delta f - \langle E\partial_v\delta f\rangle\right)}_{\text{nonlinear fluctuations: } t_{\mathrm{nl}} \text{ (omit in QLT)}}\,.$$

The retained streaming and electric-response terms, coupled to Poisson's equation, give linear modes $E_k, \delta f_k \propto e^{-i\omega_r t+\gamma t}$ at fixed F. Their phase varies on τ_w and their amplitude on τ_g; these are two time scales of the same linear response.

Time scale	Physical meaning
$\tau_\mathrm{w} = \lvert\omega_r\rvert^{-1}$	Fast wave oscillation (up to a factor of 2π).
$\tau_\mathrm{g} = \gamma^{-1}$	E-folding time of a growing wave's *amplitude*, with $\gamma > 0$.
t_eq	Time for $\langle E\delta f\rangle$ in (2.3) to appreciably modify the relevant part of F. Here "eq" denotes the evolving background, not collisional thermal equilibrium.
t_nl	Time on which the nonlinear fluctuation term in (2.4) becomes important. It couples modes and invalidates their independent linear responses.

For the weakly growing Langmuir waves considered here, $\gamma \ll |\omega_r|$, so $\tau_\mathrm{w} \ll \tau_\mathrm{g}$. Also require $\tau_\mathrm{w} \ll t_\mathrm{eq}$: a wave sees an almost fixed background during one oscillation.

Average over fast oscillations without erasing the slow evolution of either the background or the wave intensity. The averaging window must therefore satisfy

$$\boxed{\tau_\mathrm{w} \ll \Delta t \ll \min(t_\mathrm{eq}, \tau_\mathrm{g}).} \tag{5.11}$$

The growth time and t_eq need not be equal: γ depends on the resonant slope of F, whereas the rate of change of F also depends on the *wave intensity*. Initially weak waves can grow while barely changing the distribution.

5.4 The central QLT ordering and what it permits

Assume perturbatively small fluctuations, $|\delta f| \ll F$. For QLT to describe relaxation of the unstable background, their mean feedback must act before the neglected nonlinear fluctuation dynamics becomes important:

$$\boxed{\tau_\mathrm{w} \ll t_\mathrm{eq} \ll t_\mathrm{nl}.} \tag{5.12}$$

Short-time linear theory applies while $t \ll t_\mathrm{eq}, t_\mathrm{nl}$. QLT extends the calculation to $t \sim t_\mathrm{eq}$, but still requires $t \ll t_\mathrm{nl}$: F evolves while the fluctuations retain a linear response about the current F.

To derive the coupled system (5.10), make the following approximation to the mean and fluctuation equations:

$$(\partial_t + v\partial_z)\delta f = -\frac{q}{m}E\partial_v F, \qquad \text{linear response about the current } F, \tag{5.13}$$

$$\partial_t F = -\frac{q}{m}\partial_v \langle E\delta f\rangle, \qquad \text{slow feedback on the mean.} \tag{5.14}$$

QLT omits the *entire bracketed term* in the fluctuation equation shown above, while retaining the quadratic mean feedback. This keeps the leading linear response of the fluctuations and the leading nonzero change of F. The ordering $t_\mathrm{eq} \ll t_\mathrm{nl}$ requires that this mean change occur before the omitted fluctuation dynamics becomes important.

This ordering must survive as the waves grow and F relaxes; small initial fluctuations alone do not establish it. If $t_\mathrm{nl} \lesssim t_\mathrm{eq}$, nonlinear fluctuation interactions must be included. When $\tau_\mathrm{w} \ll t_\mathrm{nl}$ they can sometimes be treated by *weak-turbulence theory*; when $t_\mathrm{nl} \sim \tau_\mathrm{w}$ even that separation is lost [1].

5.5 Derive the wave evolution equation

Use the Fourier convention

$$E(z,t) = \sum_{k\neq 0} E_k(t)e^{ikz}, \qquad E_{-k} = E_k^*, \qquad \left\langle E^2\right\rangle = \sum_{k\neq 0}|E_k|^2. \tag{5.15}$$

Every sum includes both signed wavenumbers. Real fields have $\omega_{r,-k} = -\omega_{r,k}$ and $\gamma_{-k} = \gamma_k$.

The fluctuation equation (5.13) evolves δf_k. To determine the field as well, couple it to Poisson's equation:

$$\boxed{(\partial_t + ikv)\delta f_k = -\frac{q}{m} E_k \partial_v F, \qquad ikE_k = 4\pi q \int \delta f_k \, \mathrm{d}v.} \tag{5.16}$$

For a fixed F, this is the linear problem solved in Section 3: a normal mode has $E_k \propto e^{-i\omega_k t}$, with $\omega_k = \omega_{r,k} + i\gamma_k$ determined by $\epsilon(\omega_k, k; F) = 0$. QLT retains that mode while allowing F to evolve slowly. To leading order in this slow-background approximation, the wave amplitude obeys

$$\boxed{\frac{\mathrm{d}E_k}{\mathrm{d}t} = \left[-i\omega_{r,k}[F] + \gamma_k[F]\right] E_k.} \tag{5.17}$$

The imaginary term advances the wave phase; the real term changes its amplitude. Both coefficients are evaluated using the current distribution.

Now form the intensity $I_k = E_k E_k^*$. Differentiating and using (5.17) gives

$$\begin{aligned} \frac{\mathrm{d}I_k}{\mathrm{d}t} &= E_k^* \frac{\mathrm{d}E_k}{\mathrm{d}t} + E_k \frac{\mathrm{d}E_k^*}{\mathrm{d}t} \\ &= \left[(-i\omega_{r,k} + \gamma_k) + (i\omega_{r,k} + \gamma_k)\right] |E_k|^2 = 2\gamma_k[F] I_k. \end{aligned} \tag{5.18}$$

The phase terms cancel. This is precisely the wave-intensity equation (5.10b), the second half of the coupled QLT system. We next derive the first half: how these waves change F.

5.6 Derive the particle diffusion equation

For the dominant growing mode, the linear response (3.1), evaluated using the current F, is

$$\delta f_k \simeq -i \frac{q}{m} \frac{E_k \partial_v F}{\omega_{r,k} - kv + i\gamma_k}, \qquad \gamma_k > 0. \tag{5.19}$$

The correlation is $\langle E\delta f \rangle = \sum_k E_k^* \delta f_k$. For real fields, $E_{-k} = E_k^*$ and $\delta f_{-k} = \delta f_k^*$, so

$$E_k^* \delta f_k + E_{-k}^* \delta f_{-k} = E_k^* \delta f_k + (E_k^* \delta f_k)^* = 2\,\mathrm{Re}(E_k^* \delta f_k).$$

The imaginary parts cancel between k and $-k$. Since $\sum_k$ already includes both signs, no additional factor of 2 is needed. Using

$$\mathrm{Re}\,\frac{-i}{x + i\gamma} = -\frac{\gamma}{x^2 + \gamma^2}$$

gives

$$\langle E\delta f \rangle = -\frac{q}{m} \sum_k I_k \frac{\gamma_k}{(kv - \omega_{r,k})^2 + \gamma_k^2} \, \partial_v F. \tag{5.20}$$

Substituting into (5.14) gives the diffusion equation (5.10a), with

$$\boxed{D(v,t) = \frac{q^2}{m^2} \sum_k I_k(t) \frac{\gamma_k}{(kv - \omega_{r,k})^2 + \gamma_k^2}.} \tag{5.21}$$

D is nonnegative for this growing-wave calculation, with units of velocity squared per time. The flux $\Gamma_v = -D\partial_v F$ carries particles down the distribution gradient.

For a feature of velocity width Δv_F, a characteristic instantaneous relaxation time is

$$t_{\mathrm{eq}} \sim \frac{(\Delta v_F)^2}{D_{\mathrm{typ}}}, \tag{5.22}$$

where D_{typ} is representative of that interval. As the waves grow, D increases and this time decreases. The actual evolution follows the coupled distribution and wave equations, not a fixed diffusion time.

5.7 Which particles diffuse?

For a smooth, dense spectrum, the growing-mode limit $\gamma_k \to 0^+$ gives

$$\frac{\gamma}{x^2+\gamma^2} \longrightarrow \pi\delta(x), \qquad \boxed{D_{\rm res}(v,t) = \frac{\pi q^2}{m^2}\sum_k I_k(t)\delta(kv-\omega_{r,k}).} \tag{5.23}$$

Diffusion is concentrated at wave phase velocities. The delta function is a spectrum-level resonance rule; a single eternally coherent wave instead permits coherent particle trapping (Section 7.1). The force-correlation interpretation and its additional time separation are developed in the next subsection.

The finite-γ coefficient (5.21) was derived for growing modes. Inserting a negative damping rate into that expression would not give a valid negative diffusion coefficient; damped-wave evolution requires the causal initial-value response.

We now have the ingredients of (5.10). Given an initial distribution and a small seed wave spectrum, use F to calculate the wave growth rates, use the spectrum to calculate D, and evolve both together. Before applying this system to bump-on-tail, we make the particle interpretation of D quantitative.

5.8 Particle interpretation: from electric-field kicks to $D(v)$

The random-kick picture introduced at the beginning of this section can now be connected directly to the diffusion coefficient. For a weak, zero-mean, homogeneous and stationary electric field, evaluate the force along the unperturbed orbit $z = z_0 + vt$ over a correlation time. To leading order,

$$\Delta v(t) = \frac{q}{m}\int_0^t E(z_0+vs,s)\,\mathrm{d}s, \qquad C_v(\tau) = \langle E(z_0,0)E(z_0+v\tau,\tau)\rangle\,. \tag{5.24}$$

Squaring the time integral and using stationarity gives

$$\left\langle(\Delta v)^2\right\rangle = \frac{q^2}{m^2}\int_0^t\int_0^t C_v(s-s')\,\mathrm{d}s\,\mathrm{d}s' = \frac{2q^2}{m^2}\int_0^t (t-\tau)C_v(\tau)\,\mathrm{d}\tau. \tag{5.25}$$

The two equal triangles of the time-integration square give the factor of two. Define τ_c as the decay time of the force correlation *along a particle orbit*. For many decorrelated kicks to accumulate while F remains nearly fixed, there must be an intermediate interval

$$\boxed{\tau_c \ll t \ll t_{\rm eq}.} \tag{5.26}$$

Over this interval, the variance becomes

$$\boxed{\left\langle(\Delta v)^2\right\rangle \simeq 2D(v)t, \qquad D(v) = \frac{q^2}{m^2}\int_0^\infty C_v(\tau)\,\mathrm{d}\tau.} \tag{5.27}$$

Thus D measures the accumulated effect of correlated acceleration. The kicks must be weak enough that evaluating the force on the unperturbed orbit is accurate during each τ_c. For a self-consistent growing spectrum, the local stationary-correlation approximation also requires $\tau_c \ll \tau_{\rm g}$, so the intensity changes little during a correlated encounter.

For a Fourier component with phase $kz-\omega t$, the orbit samples the phase at the rate $kv-\omega$. Near resonance, successive kicks reinforce one another; far from resonance, their signs cancel. A finite correlation time broadens the interaction to $|kv-\omega|\tau_c \lesssim 1$, while the long-correlation limit recovers the resonant delta function in (5.23). Thus the force-correlation calculation and the distribution-function calculation are two views of the same quasilinear diffusion.

For a *prescribed* random field, use this D in the diffusion equation but do not evolve the field using (5.10b). Particle number is conserved for vanishing boundary flux, while particle kinetic energy can change through work by the externally maintained field. In the self-consistent bump-on-tail problem, by contrast, the field and distribution evolve together and share a finite energy budget. We now turn to that example.

6 Bump-on-tail relaxation: plateau and energy budget

6.1 The first equation builds a plateau

Start with the positive slope in the bump: for $k > 0$ on the positive-frequency Langmuir branch, $\gamma_k \propto F'(\omega_p/k) > 0$. The waves grow, increasing I_k and hence the resonant diffusion coefficient D.

The first equation of (5.10) then redistributes particles. Its velocity-space flux, $\Gamma_v = -D\partial_v F$, carries particles from larger to smaller values of F: the dip fills and the bump is reduced. If the wave spectrum covers a connected resonant interval, diffusion flattens that interval. A stationary state with zero particle flux requires

$$\Gamma_v = -D\partial_v F = 0 \quad \Longrightarrow \quad F(v) = F_{\rm pl} \quad (v_1 < v < v_2,\ D > 0). \tag{6.1}$$

The plateau belongs to the mean distribution F. Fine structure can remain in the full phase-space distribution, and velocities outside the resonant interval need not become Maxwellian.

Particle conservation determines the plateau height for a given interval:

$$\boxed{F_{\rm pl}(v_2 - v_1) = \int_{v_1}^{v_2} F_i(v)\,{\rm d}v, \qquad F_i(v_1) = F_i(v_2) = F_{\rm pl}.} \tag{6.2}$$

Here F_i is the initial distribution. In the idealized plateau construction, matching to the distribution outside the resonant interval, together with the equal-area condition, determines the endpoints and height.

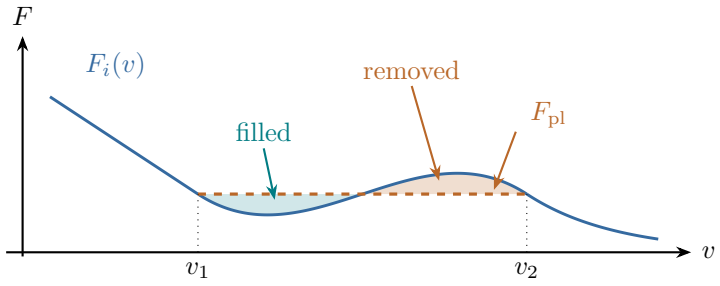


Figure 5: The plateau fills the dip and removes the bump. The two shaded areas are equal: particle number is conserved in the resonant interval.

6.2 The second equation explains saturation

As diffusion flattens the resonant slope, it also reduces the growth rate in the second equation of (5.10):

$$\partial_v F|_{v=\omega_{r,k}/k} \longrightarrow 0 \quad \Longrightarrow \quad \gamma_k[F] \longrightarrow 0 \quad \Longrightarrow \quad \frac{{\rm d}I_k}{{\rm d}t} \longrightarrow 0. \tag{6.3}$$

This is *quasilinear saturation*: the waves have removed their own drive. Their intensity can remain finite. So can D; the particle flux vanishes because the distribution has no gradient left in the resonant interval.

The coupled equations predict both the relaxation and wave growth. The condition $\gamma_k = 0$ identifies a marginal final state, but does not by itself determine the saturated intensity; that requires the initial conditions, the coupled evolution, and its conservation laws. The energy budget below supplies an additional constraint.

This ideal saturation assumes an isolated, collisionless system and a wave spectrum for which QLT remains valid. Coherent trapping can interrupt the evolution before this state is reached (Section 7.1); a plateau alone does not verify the QLT ordering [1].

6.3 Account for the energy supplied by the beam

A short moment calculation shows the direction of energy transfer:

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \int \frac{mv^2}{2}\partial_v(D\partial_v F)\,\mathrm{d}v = -m\int vD\partial_v F\,\mathrm{d}v. \tag{6.4}$$

The positive-slope resonant region at $v > 0$ loses energy. Where does it go? A Langmuir wave contains both electric energy and the oscillatory kinetic energy of the nonresonant core. For a long-wavelength, weakly growing wave, the cold-core estimate is

$$|\delta U_k| \simeq \frac{e|E_k|}{m\omega_p} \quad\Longrightarrow\quad K_{\mathrm{osc}} = \frac{mn_0}{2}\left\langle \delta U^2 \right\rangle \simeq \frac{\langle E^2\rangle}{8\pi} = U_E. \tag{6.5}$$

Thus, to this accuracy,

$$\boxed{W_{\mathrm{wave}} = U_E + K_{\mathrm{osc}} \simeq 2U_E, \qquad -\Delta K_{\mathrm{res}} \simeq \Delta W_{\mathrm{wave}}.} \tag{6.6}$$

Half of the resonant energy loss goes into electric energy and half into core oscillations in this Langmuir limit. This is not a universal partition for all plasma waves. These oscillations are also not yet irreversible thermal heating.

If K includes *all* particles, it already contains the core oscillatory energy, so total energy is $K + U_E$. If one instead separates resonant particles and the wave, use $K_{\mathrm{res}} + W_{\mathrm{wave}}$ to this order. The two accounts describe the same exchange without counting the core motion twice.

The example is now complete: a positive slope drives waves; the waves diffuse resonant particles; the slope flattens and the drive weakens. Particle conservation fixes the plateau, while energy conservation explains both the wave field and the response of the nonresonant core.

7 Applications and limits

7.1 When coherent trapping matters

In a steady wave $\phi = \phi_0\cos[k(z - v_\phi t)]$, the particle energy in the wave frame is

$$\mathcal{E}' = \frac{1}{2}m(v - v_\phi)^2 + q\phi. \tag{7.1}$$

A particle that cannot cross a potential-energy maximum follows a closed orbit in the wave frame and is trapped. Linearizing its motion about a stable potential-energy minimum gives

$$\ddot{\delta\theta} + \omega_b^2\delta\theta = 0, \qquad \boxed{\omega_b = |k|\sqrt{\frac{|q\phi_0|}{m}} = \sqrt{\frac{|q|\,|k|E_w}{m}}, \quad E_w = |k\phi_0|.} \tag{7.2}$$

This is coherent oscillation in a wave potential, rather than a sequence of independent kicks.

Define $\tau_b = \omega_b^{-1}$, the characteristic bounce time. For the random-kick picture to apply to a wave packet, a useful weak-trapping condition is

$$\boxed{\tau_c \ll \tau_b \quad\Longleftrightarrow\quad \omega_b\tau_c \ll 1.} \tag{7.3}$$

The field then loses coherence before the particle makes a substantial bounce. If $\tau_b \lesssim \tau_c$, trapped particles can remain phase-locked to the wave, so independent diffusive kicks are not an adequate description.

For a nearly monochromatic growing wave, compare the same bounce rate with the growth rate. When $\omega_b \gtrsim \gamma$, resonant particles can bounce within a growth time and trapping can alter or arrest the growth [2]. This is a different saturation mechanism from the QLT plateau. The three tests have distinct jobs: $\tau_c \ll \tau_b$ suppresses coherent trapping, $\tau_c \ll t_{\mathrm{eq}}$ permits many kicks before the background changes, and $t_{\mathrm{eq}} \ll t_{\mathrm{nl}}$ lets the mean relax before neglected nonlinear fluctuation dynamics takes over.

7.2 From velocity diffusion to space-plasma transport

The magnetized generalization of QLT allows diffusion in energy and pitch angle,

$$\mathcal{E} = \frac{mv^2}{2}, \qquad \alpha = \cos^{-1}(v_\parallel/v), \qquad D_{\alpha\alpha} \simeq \frac{\langle(\Delta\alpha)^2\rangle_c}{2\Delta t}, \quad D_{\mathcal{E}\mathcal{E}} \simeq \frac{\langle(\Delta\mathcal{E})^2\rangle_c}{2\Delta t}. \tag{7.4}$$

The subscript c denotes centered variances, measured over times longer than the correlation time but shorter than the distribution-evolution time. A full description also includes mixed diffusion and the correct velocity-space measure. For example, pure pitch-angle diffusion at fixed speed is

$$\partial_t F = \frac{1}{\sin\alpha}\partial_\alpha \left(\sin\alpha\, D_{\alpha\alpha}\partial_\alpha F\right). \tag{7.5}$$

Scattering into a loss cone can produce precipitation. Combining streaming with scattering leads naturally to the kinetic transport problem.

The bump-on-tail example also connects to solar-flare electrons exciting Langmuir waves. Type-III radio emission requires additional conversion into escaping electromagnetic waves [11].

Cyclotron-resonant scattering uses

$$\omega - k_\parallel v_\parallel = n\Omega, \qquad \Omega = \frac{qB}{mc}. \tag{7.6}$$

The $n = 0$ parallel electric resonance connects to our calculation. General $n \neq 0$ interactions, including those relevant to chorus and EMIC waves, require gyromotion in the fluctuating response and magnetized Vlasov QLT [8, 9]; they are not obtained from the elementary drift-kinetic equation (1.5).

For space-plasma applications, QLT answers a local interaction question: given a measured or modeled wave spectrum, which particles satisfy the resonance condition, and what pitch-angle, momentum, and mixed diffusion coefficients $D_{\alpha\alpha}$, D_{pp}, and $D_{\alpha p}$ result? For whistler-mode chorus, these coefficients describe resonant electron scattering and energization; for EMIC waves, they estimate scattering toward the loss cone; and for solar energetic particles, $D_{\mu\mu}$ from resonant Alfvénic fluctuations sets a parallel mean free path. These coefficients are inputs rather than complete population predictions. Determining how rapidly a loss cone fills, whether a MeV electron population develops, how much precipitation accumulates during an event, or what SEP arrival profiles and anisotropies are observed requires a Fokker–Planck or focused-transport equation for $F(\boldsymbol{x}, p, \mu, t)$, together with sources, escape, focusing, cooling, and boundary conditions. Strong coherent wave packets, field-line wandering, and the difficulty of scattering through $\mu \simeq 0$ also mark situations in which a simple QLT closure may be incomplete.

Recommended reading

Kulsrud, *Plasma Physics for Astrophysics*, Chapter 10, especially Sections 10.4–10.7. Use this for Landau resonance, the physical picture of Landau damping, different types of wave–particle resonance, and the drift-kinetic equation [3].

Parra, *Drift Kinetics.* Use this for a systematic derivation of drift kinetics from the Vlasov equation. The notes introduce guiding-center coordinates, order the problem in $\rho/L \ll 1$, show why the lowest-order distribution is gyrotropic, and derive the guiding-center phase-space equation and its Jacobian [4]. Online lecture notes.

Dodin, *Plasma Waves and Instabilities*, Lecture 15, especially Sections 15.1–15.3. Use this for quasilinear theory: the distinction between coherent trapping by one wave and statistical diffusion by many waves, the velocity-space diffusion coefficient, conservation laws, and bump-on-tail relaxation [5]. Download the lecture notes.

References

[1] A. A. Schekochihin, *Lectures on Kinetic Theory and Magnetohydrodynamics of Plasmas*, Oxford MMathPhys lecture notes (version compiled 16 August 2026). Lecture notes.

[2] B. D. Fried, C. S. Liu, R. W. Means, and R. Z. Sagdeev, "Nonlinear Evolution and Saturation of an Unstable Electrostatic Wave," UCLA Report PPG-93 (1971). Report.

[3] R. M. Kulsrud, *Plasma Physics for Astrophysics* (Princeton University Press, 2005).

[4] F. I. Parra, *Drift Kinetics*, Oxford lecture notes (20 January 2019). Lecture notes.

[5] I. Y. Dodin, *Plasma Waves and Instabilities*, Princeton University AST 553 lecture notes, Lecture 15. Lecture notes.

[6] K. G. Klein and G. G. Howes, "Measuring Collisionless Damping in Heliospheric Plasmas using Field–Particle Correlations," *Astrophysical Journal Letters* **826**, L30 (2016). Article.

[7] C. H. K. Chen, K. G. Klein, and G. G. Howes, "Evidence for electron Landau damping in space plasma turbulence," *Nature Communications* **10**, 740 (2019). doi:10.1038/s41467-019-08435-3.

[8] C. F. Kennel and F. Engelmann, "Velocity Space Diffusion from Weak Plasma Turbulence in a Magnetic Field," *Physics of Fluids* **9**, 2377–2388 (1966). doi:10.1063/1.1761629.

[9] A. J. Brizard and A. A. Chan, "Hamiltonian formulations of quasilinear theory for magnetized plasmas," *Frontiers in Astronomy and Space Sciences* **9**, 1010133 (2022). doi:10.3389/fspas.2022.1010133.

[10] A. A. Schekochihin, J. T. Parker, E. G. Highcock, P. J. Dellar, W. Dorland, and G. W. Hammett, "Phase mixing versus nonlinear advection in drift-kinetic plasma turbulence," *Journal of Plasma Physics* **82**, 905820212 (2016). doi:10.1017/S0022377816000374.

[11] H. A. S. Reid and H. Ratcliffe, "A review of solar type III radio bursts," *Research in Astronomy and Astrophysics* **14**, 773–804 (2014). Article.