P122 | Advanced Topics in Plasma Physics

Fokker–Planck Transport and Particle Acceleration

From the kicks on one particle to the evolution of a population

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Topic 6 ended with one result: many weak, decorrelated resonant kicks make a particle random-walk in velocity, and the mean distribution obeys $\partial_t F = \partial_v(D\,\partial_v F)$. That calculation lived in a periodic box with one velocity coordinate and no sources, losses, or boundaries. Real energetic-particle problems have all of these. The question of these notes is: *once the microscopic physics has supplied drift and diffusion coefficients, how does a whole population evolve?*

We first build the Fokker–Planck equation from the statistics of small random steps. We then apply it to the three variables of a magnetized energetic particle, in the order in which they usually relax: its *direction* (pitch angle), its *position* along the field, and its *energy*. One idea organizes everything. Drift kinetics averaged over the fast gyrophase; quasilinear theory averaged over fast wave phases. Here we keep doing the same thing: the fastest variable is averaged away, and the slower ones inherit a drift and a diffusion built from it.

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1 From quasilinear diffusion to population transport

Topic 6 followed particles interacting with weak waves. Once the wave phase decorrelates, successive resonant kicks add like a random walk. In the simplest one-dimensional example,

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial v}\left(D(v)\frac{\partial F}{\partial v}\right), \qquad J_v = -D(v)\frac{\partial F}{\partial v}. \tag{1.1}$$

This is already a Fokker–Planck equation. QLT has supplied a particular velocity-space flux. The next question is broader: how does an entire population evolve when particles can also stream through space, focus in a nonuniform magnetic field, change energy, escape, or enter through a source?

Three plasma problems motivate the extension:

Physical problem	What must be followed
Solar energetic particles	streaming, magnetic focusing, and pitch-angle scattering
Radiation-belt precipitation	pitch-angle diffusion together with the atmospheric loss cone
Particle acceleration	momentum diffusion, systematic gains or losses, injection, and escape

The role of the Fokker–Planck framework is to put these processes into one conservation law. The coefficients still have to come from physics—for example QLT, collisions, particle orbits, simulations, or observations.

1.1 The plan: a ladder of fast variables

A magnetized energetic particle has several clocks. A useful baseline ordering is

$$\underbrace{\Omega^{-1}}_{\text{gyration}} \ll \underbrace{\tau_c}_{\text{wave phase}} \ll \underbrace{\tau_{\text{sc}}}_{\text{direction}} \ll \underbrace{L/v}_{\text{position}} \ll \underbrace{\tau_{\text{acc}},\ \tau_{\text{esc}}}_{\text{energy}}. \tag{1.2}$$

Here τ_{sc} is the time to randomize the pitch angle, L is the size of a structure along the field, and τ_{acc} and τ_{esc} are the times to change the energy appreciably or to leave the region. Figure 1 shows the lecture as one walk down this ladder. The ordering is a guide; later sections also discuss what happens when neighboring timescales are not well separated.

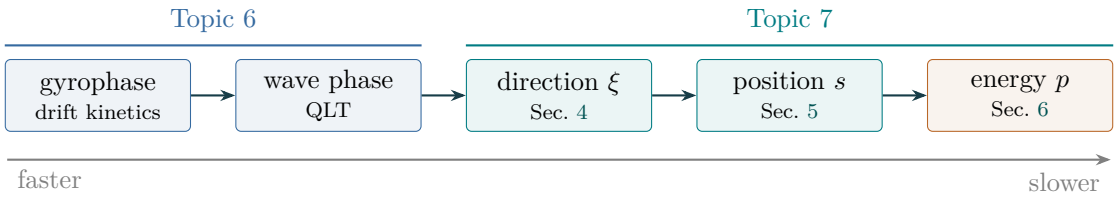


Figure 1: The organizing idea. Each variable is averaged once it has relaxed; the variables to its right then evolve by a slow drift and diffusion whose coefficients come from the averaging.

1.2 The timescale picture

Let τ_c be the time over which the fluctuating force sampled by a particle remains correlated. During one interval τ_c, a weak interaction gives a small pitch-angle kick $|\delta\xi_c| \ll 1$. After the force decorrelates, the next kick has an unrelated phase. Many small kicks therefore accumulate as

$$\left\langle (\Delta\xi)^2 \right\rangle \sim \frac{t}{\tau_c}\left\langle (\delta\xi_c)^2 \right\rangle. \tag{1.3}$$

The scattering time $\tau_{\rm sc}$ is the time for these small deflections to produce an order-unity change of direction:

$$\boxed{\tau_{\rm sc} \sim \frac{\tau_c}{\langle(\delta\xi_c)^2\rangle} \gg \tau_c.} \tag{1.4}$$

Thus $\tau_{\rm sc}$ is a cumulative weak-scattering time, not the waiting time for one discrete collision. Before this time, the particle remembers its direction and can move nearly ballistically. After many scattering times, direction can often be averaged away and spatial diffusion becomes useful. This transition from direction to position, and then to energy, provides the storyline of these notes.

2 From random kicks to the Fokker–Planck equation

We first use one general coordinate q. It could later become position s, pitch-angle cosine ξ, momentum p, or a velocity component.

Choose a time interval Δt that is long compared with the force-correlation time but short enough that the distribution has barely changed. For one particle, define

$$\Delta q \equiv q(t+\Delta t) - q(t). \tag{2.1}$$

Now consider many particles that begin at the same q. Their mean displacement and their spreading define

$$\boxed{A(q) \equiv \frac{\langle \Delta q\rangle_q}{\Delta t}, \qquad D(q) \equiv \frac{\mathrm{Var}(\Delta q)_q}{2\Delta t} \simeq \frac{\langle(\Delta q)^2\rangle_q}{2\Delta t}.} \tag{2.2}$$

$A(q)$ tells us how fast the center of the population moves through q-space. $D(q)$ tells us how fast the population spreads around that center. Here we use the simple local picture in which D changes little across one small step.

The corresponding particle flux is

$$\boxed{J_q = A(q)F - D(q)\frac{\partial F}{\partial q}.} \tag{2.3}$$

The first term is advection by the mean displacement; the second is diffusion. Particle conservation gives $\partial_t F = -\partial_q J_q$, and therefore

$$\boxed{\frac{\partial F}{\partial t} = -\frac{\partial}{\partial q}\left[A(q)F\right] + \frac{\partial}{\partial q}\left[D(q)\frac{\partial F}{\partial q}\right].} \tag{2.4}$$

This is the one-dimensional Fokker–Planck equation. Nothing new was added between Equations (2.3) and (2.4): the latter is simply the conservation law for the former.

A reflecting boundary has $J_q = 0$. An absorbing boundary removes particles that reach it. These boundaries will become important for loss cones and escape.

2.1 How QLT fits inside this equation

Topic 6 gave the Fick-law flux $J_v = -D(v)\partial_v F$. This is exactly the diffusive term in (2.3), with no additional systematic advection. Therefore

$$\boxed{A(v) = 0.} \tag{2.5}$$

Equation (2.4) then reduces directly to Equation (1.1). QLT is therefore one particular Fokker–Planck model: it supplies the diffusion coefficient and the associated flux for a specified set of weak waves.

For a magnetized energetic particle we will use three slow coordinates:

Coordinate	Coefficient	Meaning
$\xi = v_\parallel/v$	$D_{\xi\xi}$	randomization of direction
s	$\kappa_\parallel$	spreading along the magnetic field
p	D_{pp}	stochastic change of momentum

We now discuss these in that order.

3 A transport equation for magnetized particles

After averaging over gyrophase, describe a particle by

$$f = f(s, p, \xi, t), \qquad \xi \equiv \cos\alpha = \frac{v_\parallel}{v}. \tag{3.1}$$

Here s is distance along the mean magnetic field, p is momentum magnitude, and ξ specifies direction. A useful transport equation is

$$\boxed{\frac{\partial f}{\partial t} + \xi v \frac{\partial f}{\partial s} + \frac{(1-\xi^2)v}{2L}\frac{\partial f}{\partial \xi} = \frac{\partial}{\partial \xi}\left(D_{\xi\xi}\frac{\partial f}{\partial \xi}\right) + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(D_{pp}\frac{\partial f}{\partial p} - \dot{p}_{\rm sys} f\right)\right] + Q - \frac{f}{\tau_{\rm esc}}.} \tag{3.2}$$

Advection versus diffusion. In velocity space, magnetic focusing advects f in direction through $\dot{\xi}_{\rm foc}\,\partial_\xi f$, while $-p^{-2}\partial_p(p^2\dot{p}_{\rm sys}f)$ advects particles in momentum magnitude. The terms containing $D_{\xi\xi}$ and D_{pp} describe diffusion in direction and momentum, respectively; $\xi v\,\partial_s f$ is spatial advection. Systematic momentum gain ($\dot{p}_{\rm sys} > 0$) can arise from coherent electric acceleration; systematic loss ($\dot{p}_{\rm sys} < 0$) includes synchrotron or inverse-Compton cooling and Coulomb drag. A general coherent electric force also changes pitch angle and requires an additional angular drift.

The equation is a menu rather than a model that must always be used in full:

Term	Physical role
$\xi v\,\partial_s f$	streaming along the magnetic field
$[(1-\xi^2)v/(2L)]\,\partial_\xi f$	magnetic focusing
$D_{\xi\xi}$	pitch-angle scattering
D_{pp}	stochastic energization
$\dot{p}_{\rm sys}$	coherent acceleration or energy loss
Q **and** $\tau_{\rm esc}$	injection and escape

Each application keeps only the terms that matter. We first follow direction, then position, and finally energy.

4 Direction: focusing and pitch-angle scattering

4.1 Magnetic focusing

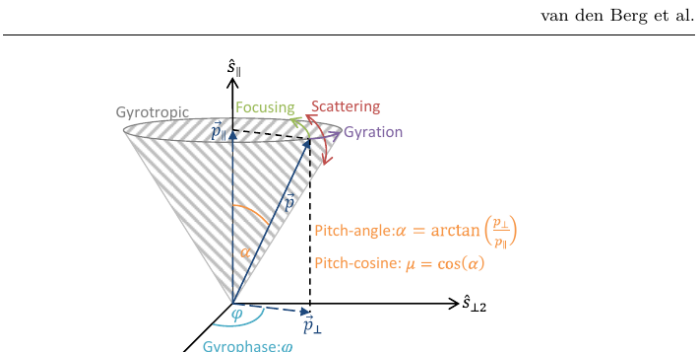


Figure 2: Pitch angle α, pitch-angle cosine $\xi = \cos\alpha$, gyrophase, focusing, and scattering. The figure uses μ for the pitch-angle cosine; in these notes μ is reserved for the magnetic moment. Cropped from Fig. 5 of van den Berg, Strauss, and Effenberger [4].

In a static magnetic field, the speed and the magnetic moment $\mu = mv_\perp^2/(2B)$ are conserved. If a particle moves into a weaker field, $v_\perp$ decreases and $|v_\parallel|$ increases. The particle becomes more field aligned. In terms of ξ,

$$\boxed{\dot{\xi}_{\rm foc} = \frac{(1-\xi^2)v}{2L}, \qquad \frac{1}{L} \equiv -\frac{\partial \ln B}{\partial s}.} \tag{4.1}$$

The focusing time is roughly L/v. A rapidly diverging field therefore produces a strongly beamed distribution.

4.2 Pitch-angle scattering

Resonant waves give many small changes of ξ. Their ensemble effect is

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial}{\partial \xi}\left(D_{\xi\xi}\frac{\partial f}{\partial \xi}\right).} \tag{4.2}$$

A simple model is

$$D_{\xi\xi} = \frac{\nu}{2}(1-\xi^2), \qquad \tau_{\rm sc} \sim \nu^{-1}. \tag{4.3}$$

Scattering smooths angular structure and drives the distribution toward isotropy. If $\tau_{\rm sc} \ll L/v$, scattering wins and the distribution is nearly isotropic. If $\tau_{\rm sc} \gg L/v$, focusing wins and the particles remain beamed.

4.3 Loss cones: why a coefficient is not yet a lifetime

In Earth's dipole, particles with sufficiently small pitch angle reach the atmosphere instead of mirroring. This range of angles is the loss cone. Waves such as chorus or EMIC can provide $D_{\xi\xi}$ through QLT, but $D_{\xi\xi}$ alone does not determine the lifetime. One must also impose an absorbing boundary at the loss cone and calculate the flux into it. QLT supplies the local scattering rate; the Fokker–Planck boundary-value problem supplies the precipitation rate.

5 Position: from streaming to spatial diffusion

5.1 Keep pitch angle when particles remember their direction

Keeping streaming, focusing, and scattering in (3.2) gives the focused-transport equation [3, 4]:

$$\boxed{\frac{\partial f}{\partial t} + \xi v \frac{\partial f}{\partial s} + \frac{(1-\xi^2)v}{2L}\frac{\partial f}{\partial \xi} = \frac{\partial}{\partial \xi}\left(D_{\xi\xi}\frac{\partial f}{\partial \xi}\right).} \tag{5.1}$$

Early in an SEP event, particles have not scattered enough to forget their direction. The first particles therefore arrive as a fast, beamed front. A position-only diffusion equation cannot describe this stage; the pitch-angle coordinate ξ must be retained.

5.2 After many scatterings, position diffuses

After one scattering time, a particle travels a typical distance

$$\lambda_\parallel \sim v\tau_{\rm sc}. \tag{5.2}$$

After many such steps, its position performs a random walk. The corresponding parallel diffusion coefficient is

$$\boxed{\kappa_\parallel \sim \frac{v^2\tau_{\rm sc}}{3} = \frac{v\lambda_\parallel}{3}, \qquad \frac{\partial F}{\partial t} = \frac{\partial}{\partial s}\left(\kappa_\parallel \frac{\partial F}{\partial s}\right).} \tag{5.3}$$

The factor $1/3$ comes from sharing an isotropic velocity among three spatial directions. The physical transition is

$$\left\langle (\Delta s)^2 \right\rangle \sim \begin{cases} v^2 t^2, & t \ll \tau_{\rm sc} \quad \text{(ballistic)}, \\ 2\kappa_\parallel t, & t \gg \tau_{\rm sc} \quad \text{(diffusive)}. \end{cases} \tag{5.4}$$

This explains why an SEP event can begin with a sharp, anisotropic arrival and later develop a broad, slowly varying tail.

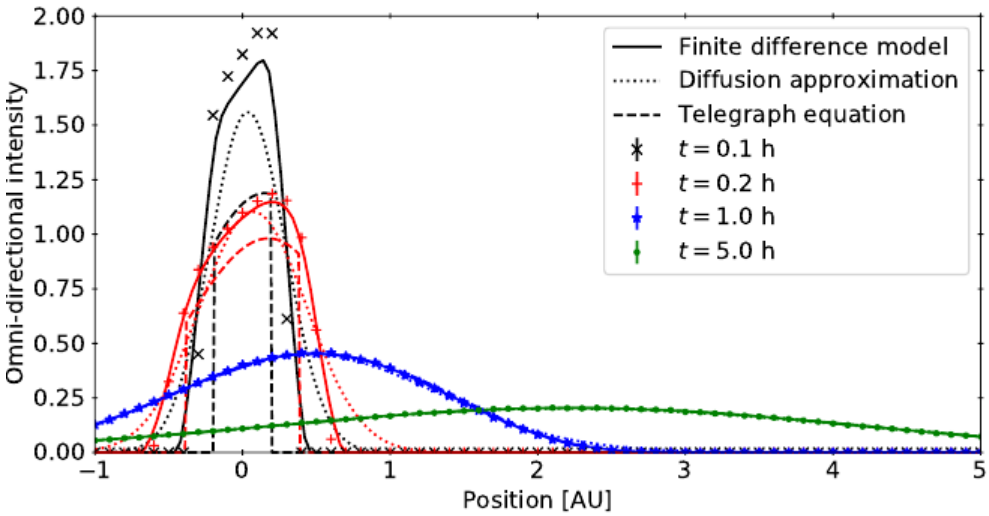


Figure 3: Focused transport compared with diffusion-based approximations. At early times the distribution retains a propagating front; diffusion becomes useful only after scattering has reduced the anisotropy. Upper panel cropped from Fig. 7 of [4].

5.3 The Parker equation: a large-scale endpoint

When the distribution is nearly isotropic, cosmic-ray transport is often written as [5]

$$\boxed{\frac{\partial F}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}F = \boldsymbol{\nabla}\cdot(\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}F) + \frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{U})\,p\frac{\partial F}{\partial p} + Q.} \tag{5.5}$$

The terms describe advection with the plasma, diffusion relative to it, and adiabatic energy change. Expansion cools the particles; compression energizes them. At a shock, the compression term leads toward first-order Fermi acceleration.

6 Energy: stochastic acceleration

Once pitch angle has been randomized, we can ask how the momentum magnitude changes.

6.1 Second-order Fermi acceleration: a moving-mirror model

Following Fermi's original picture [1], represent a moving magnetic irregularity by a mirror traveling at speed $u \ll c$. Let

$$\beta \equiv \frac{u}{c}, \qquad \chi \equiv \cos\theta, \tag{6.1}$$

where θ is the angle between the incoming particle and the mirror velocity. Thus $\chi = -1$ is a head-on encounter and $\chi = +1$ is an overtaking encounter. We use χ here to distinguish this encounter angle from both the magnetic moment μ and the pitch-angle cosine ξ.

For an ultra-relativistic particle, $E \simeq pc$ and $p_x = (E/c)\chi$. Transforming to the mirror frame gives

$$E' = \gamma_u E(1-\beta\chi), \qquad p'_x = \gamma_u\frac{E}{c}(\chi-\beta), \qquad \gamma_u = (1-\beta^2)^{-1/2}. \tag{6.2}$$

The reflection is elastic in this frame: $E'' = E'$ while $p''_x = -p'_x$. Transforming back gives

$$\frac{E_{mf}}{E} = \gamma_u^2(1+\beta^2-2\beta\chi), \qquad \frac{\Delta E}{E} \simeq -2\beta\chi + 2\beta^2. \tag{6.3}$$

A head-on encounter gains energy; an overtaking encounter loses energy.

The directions are initially isotropic, so $P_0(\chi) = 1/2$. Actual encounters are weighted by the relative speed,

$$v_{mrel} \simeq c(1-\beta\chi). \tag{6.4}$$

Head-on encounters therefore occur slightly more often, and the normalized distribution among encounters is

$$P_{\rm enc}(\chi) = \frac{1}{2}(1-\beta\chi), \qquad \langle\chi\rangle_{\rm enc} = -\frac{\beta}{3}. \tag{6.5}$$

Using this average in Equation (6.3),

$$\boxed{g \equiv \left\langle\frac{\Delta E}{E}\right\rangle = \frac{8}{3}\left(\frac{u}{c}\right)^2.} \tag{6.6}$$

The gains and losses proportional to u/c cancel in the angular average. The small excess of head-on encounters leaves the term proportional to $(u/c)^2$—the origin of the name *second-order* Fermi acceleration.

Repeated, decorrelated reflections also produce momentum diffusion. If $\tau_{\rm sc}$ is the typical time between reflections,

$$\boxed{D_{pp} \sim \frac{p^2}{\tau_{\rm sc}}\left(\frac{u}{v}\right)^2.} \tag{6.7}$$

The characteristic acceleration time is

$$\boxed{\tau_{\rm acc} \sim \frac{p^2}{D_{pp}} \sim \left(\frac{v}{u}\right)^2 \tau_{\rm sc}.} \tag{6.8}$$

Because $u \ll v$, the process is usually slow.

6.2 Escape turns multiplicative gains into a power law

Suppose each cycle multiplies the mean energy by $1+g$ and gives a particle a fixed escape probability $P_{\rm esc}$. After n cycles,

$$E_n = E_0(1+g)^n, \qquad N_n = N_0(1-P_{\rm esc})^n. \tag{6.9}$$

Eliminating n gives

$$N(>E) = N_0 \left(\frac{E}{E_0}\right)^{-a}, \qquad a = -\frac{\ln(1-P_{\rm esc})}{\ln(1+g)}. \tag{6.10}$$

The differential spectrum is one power steeper:

$$\boxed{\frac{dN}{dE} \propto E^{-s}, \qquad s = 1 + a \simeq 1 + \frac{P_{\rm esc}}{g} = 1 + \frac{t_{\rm acc}}{t_{\rm esc}}.} \tag{6.11}$$

Here $t_{\rm acc} \simeq \tau/g$ and $t_{\rm esc} \simeq \tau/P_{\rm esc}$ for a cycle time τ. Fast acceleration and slow escape produce a hard spectrum; rapid escape produces a steep spectrum. The power law comes from a fixed fractional gain together with a fixed escape probability.

6.3 Momentum-space transport

Including systematic gains or losses, injection, and escape gives

$$\boxed{\frac{\partial f}{\partial t} = \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(D_{pp}\frac{\partial f}{\partial p} - \dot{p}_{\rm sys} f\right)\right] + Q - \frac{f}{\tau_{\rm esc}}.} \tag{6.12}$$

Here D_{pp} describes stochastic energization, $\dot{p}_{\rm sys}$ describes coherent gain or energy loss, Q supplies particles, and $\tau_{\rm esc}$ removes them. The spectrum is determined by the competition among acceleration, cooling, and escape.

The continuous equation is the Fokker–Planck version of the toy model. When $D_{pp} \propto p^2$ and the escape time is momentum independent, the coefficients introduce no preferred momentum scale, so a steady power law is natural. Cooling ends the tail near the momentum where

$$\tau_{\rm acc}(p) \sim \tau_{\rm loss}(p). \tag{6.13}$$

A spectrum alone does not uniquely identify the acceleration mechanism. Different combinations of drift, diffusion, loss, and escape can produce similar distributions. Timing, anisotropy, radiation, or particle trajectories provide the additional information.

7 Where the diffusion picture fails

The Fokker–Planck picture assumes that each coarse-grained step is small and that the particle has lost memory of earlier steps.

Coherent trapping. A particle trapped in a long-lived wave, island, or plasmoid can remain correlated with the same structure for a long time. Its energy change is then a coherent history rather than a sequence of independent kicks. A diffusion coefficient measured over different time intervals may not converge.

Large jumps. A single X-point encounter or strong electric-field pulse may change the energy by an amount comparable with the width of the distribution. Such events cannot be represented by only a mean and a variance.

An incomplete state. Sometimes the process becomes Markovian only after another variable is retained. Position alone is insufficient before pitch angle has randomized; adding ξ gives the focused-transport equation. In strongly turbulent or reconnecting plasmas, one may similarly need information about which structure or acceleration mechanism is acting on a particle.

These are limits of a particular reduced description, not failures of kinetic physics. They tell us which information was averaged away too early.

8 Takeaways

1. QLT already gives a special Fokker–Planck flux. The general framework adds other coordinates, systematic motion, boundaries, sources, losses, and escape.
2. $A(q)$ describes the motion of the center of a population; $D(q)$ describes its spreading. Their flux plus particle conservation gives the Fokker–Planck equation.
3. Pitch-angle scattering is the cumulative result of many small kicks. Before directions randomize, use focused transport; afterward, spatial diffusion becomes useful.
4. Momentum diffusion describes stochastic acceleration. The competition among acceleration, cooling, and escape determines power laws and cutoffs.
5. Coherent trapping, large jumps, or missing state variables can invalidate the diffusion approximation.

Recommended reading

- Fermi (1949), *On the Origin of the Cosmic Radiation*: the original moving-magnetic-cloud model for second-order acceleration. https://doi.org/10.1103/PhysRev.75.1169
- van den Berg, Strauss, and Effenberger (2020), *A Primer on Focused Solar Energetic Particle Transport*: focused transport, SEP observations, and the transition toward diffusion. https://arxiv.org/abs/2012.07570
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- Parker (1965): the classical large-scale cosmic-ray transport equation.

- Blandford and Eichler (1987): particle acceleration and shock applications.
- Gardiner, *Stochastic Methods*, Chs. 4–5: optional reference for the formal drift–diffusion derivation.

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