P122 | Advanced Topics in Plasma Physics

Mean-field Dynamo and MRI

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Summary

Magnetic fields can acquire coherent structure on scales larger than the turbulent motions that sustain them. We derive the mean induction equation and interpret the α, β, and Ω effects as field regeneration, turbulent diffusion, and winding by differential rotation. The poloidal–toroidal regeneration loop connects these effects to solar magnetic cycles and butterfly diagrams. A simple dynamo-wave model illustrates how this loop can produce oscillation and migration.

Accretion disks provide a second application of differential rotation. Gas must lose angular momentum to move inward, and magnetic tension couples fluid elements in a way that can destabilize an otherwise Rayleigh-stable disk. A local shearing-box model leads to the magnetorotational instability (MRI) and its growth criterion. Correlated magnetic and velocity fluctuations then connect the instability to outward angular-momentum transport, accretion, and field regeneration.

Magnetic quantities use Gaussian-cgs units throughout.

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1 From small-scale amplification to a mean field

A small-scale dynamo can increase $\langle B^2 \rangle$ while the field reverses frequently and its spatial mean remains zero. A large-scale dynamo generates structure coherent over scales larger than the characteristic turbulent driving scale. These processes can coexist; one need not finish before the other begins. The question is which inductive correlations survive an average.

Choose an average that retains the organized structure of interest. For example, averaging over azimuth preserves dependence on radius and height. Write

$$\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}', \qquad \boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}', \qquad \overline{\boldsymbol{u}'} = \overline{\boldsymbol{b}'} = 0. \tag{1.1}$$

We assume Reynolds averaging rules and that the average commutes with the derivatives below. A mean need not be a uniform field or a whole-volume average.

The mean induction equation

For constant magnetic diffusivity η,

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{1.2}$$

Expand the product before averaging:

$$\begin{aligned} \overline{\boldsymbol{U} \times \boldsymbol{B}} &= \overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{U}} \times \overline{\boldsymbol{b}'} + \overline{\boldsymbol{u}'} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{u}' \times \boldsymbol{b}'} \\ &= \overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \underbrace{\overline{\boldsymbol{u}' \times \boldsymbol{b}'}}_{\boldsymbol{\mathcal{E}}}. \end{aligned} \tag{1.3}$$

The two cross terms vanish. The product of two fluctuations need not vanish. Thus

$$\boxed{\partial_t \overline{\boldsymbol{B}} = \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \boldsymbol{\mathcal{E}}) + \eta \nabla^2 \overline{\boldsymbol{B}}.} \tag{1.4}$$

The turbulent electromotive force $\boldsymbol{\mathcal{E}}$ describes induction by correlated fluctuations. Its *curl* changes the mean field: a spatially uniform EMF has no local induction effect. An EMF can regenerate, transport, or smooth the mean pattern; its existence alone does not guarantee growth.

> The mean equation is exact under these averaging assumptions. The difficult step is to relate $\boldsymbol{\mathcal{E}}$ to the mean field: this is the closure problem.

2 The alpha–beta expansion

For a mean field that varies slowly across a turbulent eddy, a **gradient expansion** of the linear turbulent response retains the local field and its first spatial derivatives:

$$\boxed{\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial_k\overline{B}_j + \cdots .} \tag{2.1}$$

In the simplest homogeneous, isotropic model, this reduces to

$$\boxed{\boldsymbol{\mathcal{E}} \simeq \alpha_{\rm dyn}\overline{\boldsymbol{B}} - \beta\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}.} \tag{2.2}$$

The α term represents regeneration of the mean field by correlated turbulent motions. Since $\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} \propto \overline{\boldsymbol{J}}$, the β term has the same current-proportional form as the resistive contribution to induction. For constant $\beta > 0$, it contributes $+\beta\nabla^2\overline{\boldsymbol{B}}$, smoothing the mean field just as Ohmic diffusion does.

With $\eta_T = \eta + \beta$ constant, the mean induction equation is

$$\partial_t\overline{\boldsymbol{B}} = \underbrace{\boldsymbol{\nabla}\times(\overline{\boldsymbol{U}}\times\overline{\boldsymbol{B}})}_{\text{mean flow, including }\Omega} + \underbrace{\boldsymbol{\nabla}\times(\alpha_{\rm dyn}\overline{\boldsymbol{B}})}_{\alpha\text{: regeneration}} + \underbrace{\eta_T\nabla^2\overline{\boldsymbol{B}}}_{\text{Ohmic + turbulent diffusion}} . \tag{2.3}$$

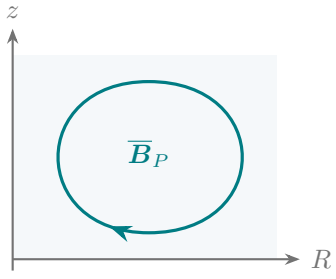
The Ω effect belongs to the mean-flow term. We use this scalar closure as a simple model; its coefficients need not be scalars in anisotropic turbulence.

Poloidal and toroidal are directions, not different kinds of field

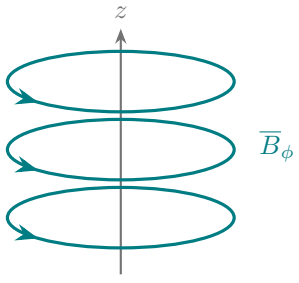
Relative to an axis of rotation, an axisymmetric field decomposes as

$$\overline{\boldsymbol{B}} = \underbrace{\overline{B}_R\hat{\boldsymbol{R}} + \overline{B}_z\hat{\boldsymbol{z}}}_{\overline{\boldsymbol{B}}_P\ \text{(poloidal)}} + \underbrace{\overline{B}_\phi\hat{\boldsymbol{\phi}}}_{\overline{\boldsymbol{B}}_T\ \text{(toroidal)}} . \tag{2.4}$$

Poloidal field lies in meridional planes containing the axis. Toroidal field points around the axis. In a star the poloidal components are usually written (B_r, B_θ); in a disk they are (B_R, B_z). A field line can have both components.



Poloidal: in an R–z plane



Toroidal: around the axis

Figure 1: The standard axisymmetric decomposition. The left panel shows a representative poloidal loop in one meridional plane; the right shows toroidal loops in perspective. These are geometric sketches, not a particular equilibrium solution.

3 The Omega effect: winding poloidal field into toroidal field

Take an axisymmetric mean field and purely rotational mean flow,

$$\overline{\boldsymbol{U}} = R\Omega(R,z)\hat{\boldsymbol{\phi}}. \tag{3.1}$$

The toroidal part of ideal induction gives the winding term

$$\boxed{\partial_t \overline{B}_\phi\Big|_\Omega = R\overline{\boldsymbol{B}}_P \cdot \boldsymbol{\nabla}\Omega = R\left(\overline{B}_R\partial_R\Omega + \overline{B}_z\partial_z\Omega\right).} \tag{3.2}$$

Different parts of a poloidal field line are carried around the axis at different angular velocities. The line develops an azimuthal component. The relevant gradient is the one *along the poloidal field*: uniform rotation gives no winding, and neither does a field tangent to surfaces of constant Ω.

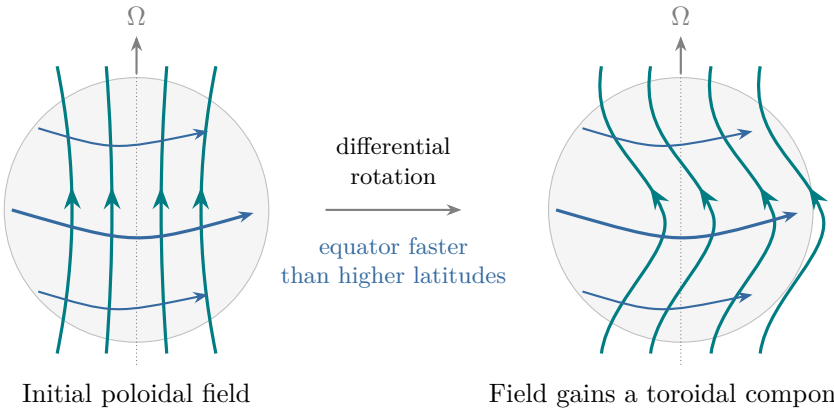


Figure 2: The familiar stellar winding picture: teal lines trace field, blue arrows show rotation. The equatorial region advances farther in azimuth. Conceptual redraw of the Ω-effect geometry in Rincon (2019), Fig. 26; line shapes and amplitudes are illustrative.

Winding alone gives linear-in-time amplification if the poloidal source is fixed. It does not regenerate that source. A self-sustained dynamo needs a return path and must overcome losses.

4 The alpha effect: correlated twisting and field regeneration

Fluctuating motions bend a mean field and produce magnetic fluctuations. If the motions have a statistical preference for one handedness, their inductive contributions need not cancel. The simplest representation is $\boldsymbol{\mathcal{E}}_\alpha = \alpha_{\mathrm{dyn}}\overline{\boldsymbol{B}}$, parallel or antiparallel to the mean field. It is the *curl of this EMF* that changes the field direction and amplitude.

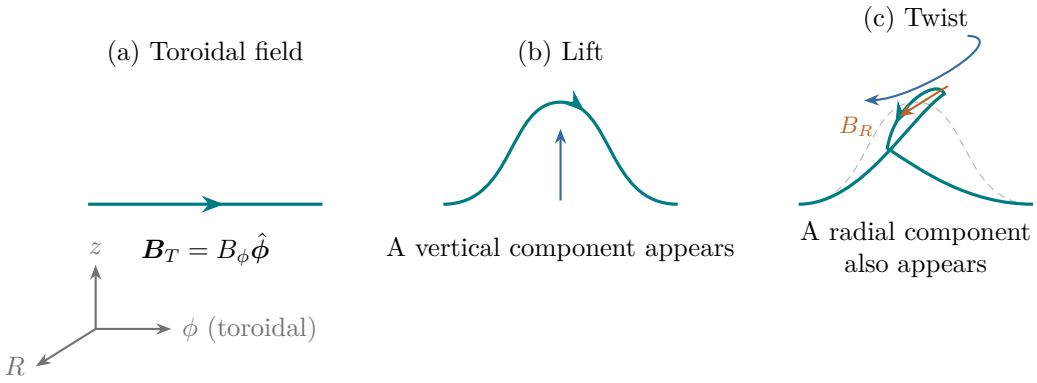


Figure 3: A local Parker-loop teaching sketch. A toroidal segment is lifted and twisted, acquiring poloidal components. In (c), gray dashes show the untwisted loop and orange marks the radial component. An ensemble with a preferred handedness can contribute coherently to the EMF; a single deformed line is not by itself a mean-field dynamo. See Rincon (2019), Fig. 27 and Section 4.2.2 for the standard picture. Internal twist and the helicity budget are not shown.

Why handedness and averaging matter

The scalar α_{dyn} changes sign under reflection. Equal populations of mirror-related motions cancel this contribution. Rotation together with stratification can permit a preferred handedness, but does not by itself specify the sign or strength of the EMF. The loop picture explains a possible correlation; computing that correlation is the closure problem.

The mean field may be axisymmetric while the motions and fluctuating fields that regenerate it are three-dimensional and non-axisymmetric. Mean-field theory does not assume a purely axisymmetric full dynamo.

5 Beta, and the competition between regeneration and losses

For positive constant β, the turbulent EMF adds $\beta\nabla^2\overline{\boldsymbol{B}}$ to the mean induction equation, smoothing gradients just as Ohmic diffusion does. This weakening of the organized mean pattern can coexist with amplification of fluctuating magnetic energy.

Closing the poloidal–toroidal regeneration loop

For axisymmetric mean fields, write $\overline{\boldsymbol{B}} = \overline{\boldsymbol{B}}_P + \overline{\boldsymbol{B}}_T$ and $\overline{\boldsymbol{U}} = \overline{\boldsymbol{U}}_P + \overline{\boldsymbol{U}}_T$, where $\overline{\boldsymbol{U}}_P$ is meridional circulation and $\overline{\boldsymbol{U}}_T = R\Omega\hat{\boldsymbol{\phi}}$ is rotation. With constant $\eta_T = \eta + \beta$, the two field equations are

$$\partial_t \overline{\boldsymbol{B}}_P = \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}}_P \times \overline{\boldsymbol{B}}_P) + \underbrace{\boldsymbol{\nabla} \times (\alpha_{\rm dyn} \overline{\boldsymbol{B}}_T)}_{\alpha\text{: } T\to P} + \eta_T \nabla^2 \overline{\boldsymbol{B}}_P, \tag{5.1}$$

$$\partial_t \overline{\boldsymbol{B}}_T = \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}}_P \times \overline{\boldsymbol{B}}_T) + \underbrace{\boldsymbol{\nabla} \times (\overline{\boldsymbol{U}}_T \times \overline{\boldsymbol{B}}_P)}_{\Omega\text{: } P\to T} + \boldsymbol{\nabla} \times (\alpha_{\rm dyn} \overline{\boldsymbol{B}}_P) + \eta_T \nabla^2 \overline{\boldsymbol{B}}_T. \tag{5.2}$$

The Ω term in the second equation winds poloidal field into toroidal field; the α term in the first regenerates poloidal field from toroidal field. Together they close the $\alpha\Omega$ **regeneration loop**. Meridional circulation transports the fields, while diffusion acts on both. Alpha can also generate toroidal field: if it drives both conversions, the model is an α^2 dynamo.

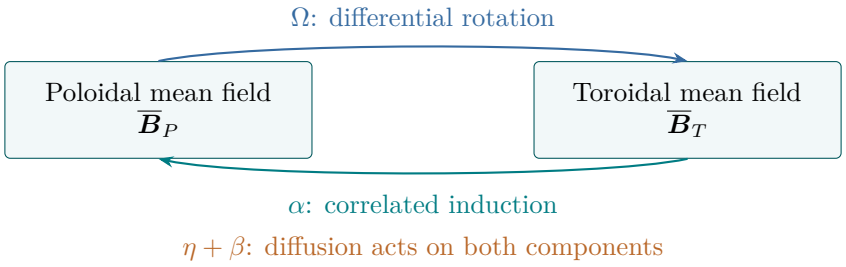
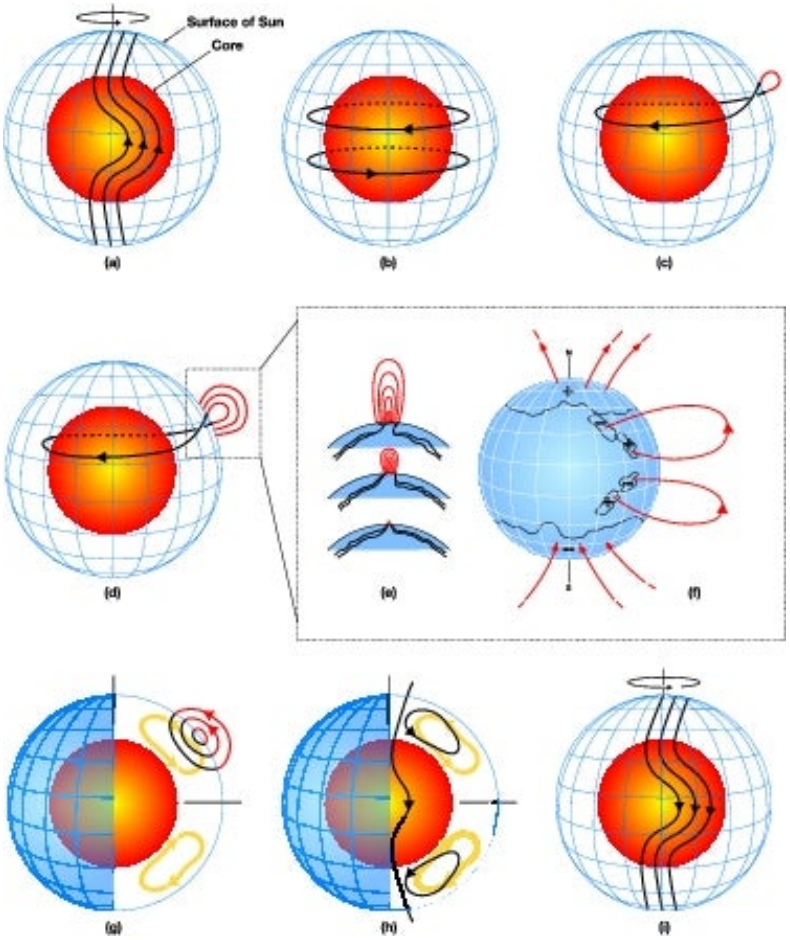


Figure 4: An $\alpha\Omega$ feedback diagram. Arrows identify couplings, not an automatic proof of growth. Their relative phase, the boundaries, and the losses matter.

Closing the loop supplies a regeneration mechanism; sustained growth requires it to overcome diffusion. Magnetic feedback eventually changes the flow and the transport coefficients.

The Sun: seeing the regeneration loop

The same two couplings organize the solar magnetic cycle. Follow this HAO schematic of a **Babcock–Leighton flux-transport dynamo**.



(a–b) Ω: $P \to T$.
Differential rotation winds poloidal field into toroidal bands. The equator rotates faster than the poles.

(c–f) Flux emergence.
Buoyant loops of toroidal field reach the surface as bipolar active regions. Rotation helps give them a systematic tilt.

(f–g) Regeneration: $T \to P$.
The tilt puts the two polarities at different latitudes. As regions decay, lower-latitude flux preferentially cancels across the equator; the remaining excess of the other polarity is carried poleward, rebuilding and reversing the polar field.

(h–i) Start the next cycle.
Transport carries poloidal flux back into the interior. Differential rotation winds this reversed field into toroidal field of the opposite polarity: the loop closes.

Figure 5: Solar field regeneration. Red marks the radiative interior; the blue grid is the surface, with the convection zone between them. Original schematic: HAO/NCAR, *The Sun as a Dynamo*, Fig. 1.

The connection to α. In the classical $\alpha\Omega$ model, correlated helical motions supply the $T \to P$ conversion. Here, the tilt and decay of bipolar regions supply it: this is the **Babcock–Leighton mechanism**. It fills the same regeneration role, but is not simply the local $\boldsymbol{\mathcal{E}}_\alpha = \alpha_{\rm dyn}\overline{\boldsymbol{B}}$ closure. In both pictures, differential rotation supplies the $P \to T$ leg.

The butterfly diagrams on the next page show the cycle and polarity reversals. This schematic illustrates one solar-dynamo model: the interior transport and the relative contributions of different poloidal-field sources remain subjects of research.

The solar butterfly diagrams

A regeneration model must explain both the magnetic cycle and its spatial pattern. These observed time–latitude diagrams show complementary parts of the cycle.

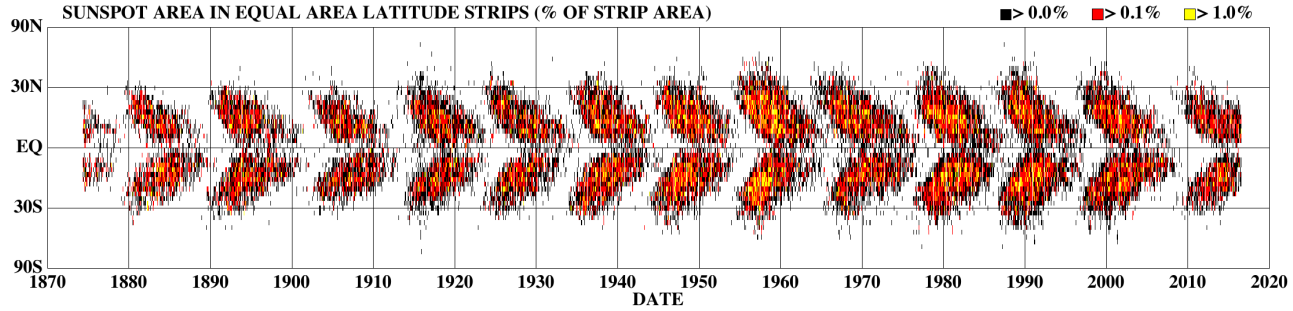


Figure 6: Sunspot butterfly diagram. New spots emerge progressively closer to the equator during each roughly 11-year activity cycle. Colors encode spot area, not magnetic polarity. Source: D. Hathaway, NASA/ARC, October 2016 compilation; upper panel of the original figure.

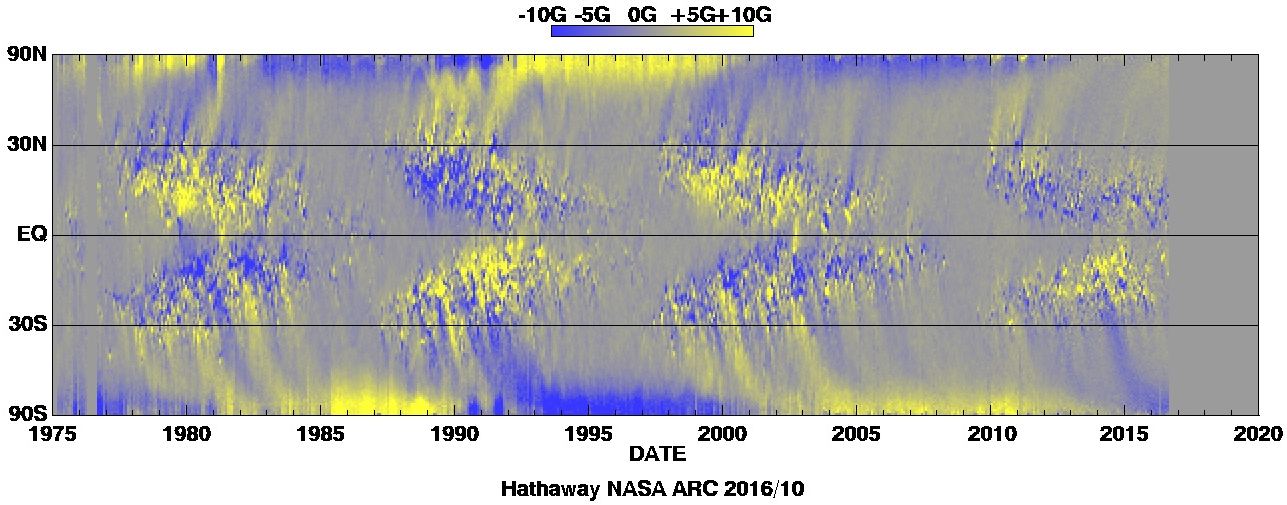


Figure 7: Magnetic butterfly diagram. The longitude-averaged surface radial field retains its sign: yellow is positive, blue negative. Poleward flux streams accompany polar-field reversals near sunspot maximum. Source: D. Hathaway, NASA/ARC, October 2016 compilation.

The polar field reverses roughly every **11 years**; a full return of magnetic polarity takes about **22 years**. The sunspot-area diagram does not distinguish field signs, so its repeated wings trace the shorter activity cycle.

Connection to mean-field dynamo. Sunspot emergence indirectly traces the interior toroidal field; surface $\overline{B}_r$ samples the poloidal field. Mean-field models predict their time–latitude patterns, including migration, reversals, and their relative timing. The local model below shows how the regeneration loop can produce an oscillating, migrating pattern.

A minimal $\alpha\Omega$ dynamo wave

Keep just the two regeneration terms. In a local patch, let B_P be the signed poloidal mean-field component along the shear gradient, B_T the toroidal component, and z distance in the perpendicular meridional direction. For constant $\alpha_{\rm dyn}$ and signed differential-rotation shear rate S, neglecting diffusion and meridional advection,

$$\boxed{\partial_t B_P = -\alpha_{\rm dyn}\partial_z B_T, \qquad \partial_t B_T = S B_P.} \tag{5.3}$$

The first equation regenerates poloidal field; the second winds it into toroidal field. Differentiate the second and substitute the first:

$$\boxed{\partial_t^2 B_T = S\partial_t B_P = -\alpha_{\rm dyn} S \partial_z B_T.} \tag{5.4}$$

This is a **dynamo-wave equation**. For $B_T \propto e^{\lambda t + ikz}$,

$$\lambda^2 = -ik\alpha_{\rm dyn} S. \tag{5.5}$$

For real nonzero $k\alpha_{\rm dyn}S$, the roots have both real and imaginary parts: amplification and oscillation occur together. Taking $k > 0$, the growing solution can be written

$$B_T \propto e^{\sigma t}\cos(kz - \omega t), \qquad \sigma = |\omega| = \sqrt{\frac{|k\alpha_{\rm dyn}S|}{2}}, \quad \omega = \mathrm{sgn}(\alpha_{\rm dyn}S)\,\sigma. \tag{5.6}$$

> At a fixed location the toroidal field repeatedly reverses sign; its crests also move. Plotted against latitude and time, such a migrating pattern gives sloping bands like butterfly wings. The direction depends on the signs of $\alpha_{\rm dyn}$ and the shear, so equatorward migration is not automatic.

The signed magnetic cycle has period $2\pi/|\omega|$; polarity reverses every half-cycle. An unsigned activity tracer can therefore repeat twice per magnetic cycle, as in the Sun's roughly 11-year activity and 22-year magnetic cycles. This model illustrates the mechanism; it does not fix those numerical periods.

Restoring constant η_T subtracts $\eta_T k^2$ from the mode's growth rate. Nonlinear feedback is needed to limit its amplitude; the linear model alone does not produce a saturated cycle.

Comparing three wave equations

For constant coefficients on an infinite or periodic domain, use the same spatial mode $\psi = \psi_0 e^{ikz+\lambda t}$, with real k. The growth rate is $\mathrm{Re}\,\lambda$ and the angular frequency is $\omega = -\,\mathrm{Im}\,\lambda$.

Equation	Time eigenvalue λ	Mode behavior
Ordinary wave $\partial_t^2\psi = c^2\partial_z^2\psi$	$\lambda = \pm ick$	Constant amplitude; $\lvert\omega\rvert \propto \lvert k\rvert$.
Free Schrödinger $i\partial_t\psi = -D\partial_z^2\psi$	$\lambda = -iDk^2$	Constant amplitude; $\omega = Dk^2$ (dispersive).
Dynamo wave $\partial_t^2\psi = C\partial_z\psi$	$\lambda = \pm\sqrt{iCk}$	Growing/decaying oscillations; $\lvert\omega\rvert \propto \lvert k\rvert^{1/2}$.

Here $D = \hbar/(2m) > 0$ and $C = -\alpha_{\rm dyn}S$ is real. The key difference is $\partial_z \to ik$ versus $\partial_z^2 \to -k^2$: together with the equation's signs and factors of i, these determine whether λ is purely imaginary or has both real and imaginary parts.

6 Astrophysical applications: what supplies the motion?

The formalism above does not prescribe how turbulence is driven. That question becomes essential when applying it to an astrophysical object.

- **Stars:** convection, rotation, and differential rotation provide motions and induction. The solar example in Fig. 5 shows how winding, flux emergence, and transport can close a poloidal–toroidal regeneration loop.
- **Galaxies:** turbulent gas motions and differential rotation can contribute to an organized field on scales larger than the turbulent forcing. Outflows, geometry, and magnetic feedback also affect the resulting structure.
- **Accretion disks:** differential rotation stores free energy. In sufficiently ionized disks, the MRI can generate motions and magnetic fluctuations, linking field regeneration to angular-momentum transport.

We now follow the disk example. MRI is an instability of a magnetized rotating flow; mean-field dynamo theory describes the evolution of its organized field. They address connected but distinct questions.

Why an accretion disk needs angular-momentum transport

An accretion disk is rotating gas that feeds a central object. Dissipation can let orbiting gas settle into a disk; efficient cooling allows it to remain thin. Accretion releases gravitational binding energy, but gas must also lose angular momentum to move inward through a sequence of nearly circular orbits.

The relation $\ell = Rv_\phi = R^2\Omega$ is kinematic in Newtonian mechanics. The *Keplerian* expressions require stronger assumptions: a dominant central point mass, nearly circular orbits, negligible radial pressure and magnetic support, and distances outside the strongly relativistic region. Then

$$\frac{v_\phi^2}{R} = \frac{GM}{R^2} \quad \Longrightarrow \quad \Omega_K = \sqrt{\frac{GM}{R^3}}, \qquad \ell_K = \sqrt{GMR}. \tag{6.1}$$

Angular velocity decreases outward, while specific angular momentum increases. A parcel moving from R to $R/4$ must lose half its specific angular momentum. Central gravity maintains an orbit but supplies no torque about the center; cooling alone does not solve this problem.

Internal stresses transfer angular momentum outward, allowing some gas to accrete. A magnetized wind can instead remove angular momentum from the disk. We focus on internal transport. Microscopic viscosity is generally too weak, so the physical origin of a much larger stress matters.

Why the flow does not automatically become unstable

An unmagnetized ring displaced outward conserves ℓ and has too little angular momentum for its new circular orbit. It tends to return inward. Keplerian rotation satisfies the axisymmetric centrifugal stability criterion $\mathrm{d}\ell^2/\mathrm{d}R > 0$.

> Magnetic tension changes this constraint by allowing displaced fluid elements to exchange angular momentum. This introduces the MRI, without making MRI a prerequisite for the mean-field theory above.

7 MRI: why coupling can destabilize a disk

The disk needs a stress to move angular momentum outward, but a Keplerian disk is stable to an unmagnetized radial displacement. Without a torque, a displaced fluid element conserves its specific angular momentum ℓ and executes an epicyclic oscillation. What changes if two elements can exert a torque on one another?

Two fluid elements connected by a weak spring

Imagine two elements initially at the same orbital radius, connected by magnetic tension. Displace one inward and the other outward. Since angular velocity decreases outward, the inner element moves ahead of the outer one, bending the connecting field. In a view projected onto the orbital plane, the field acts like a stretched spring:

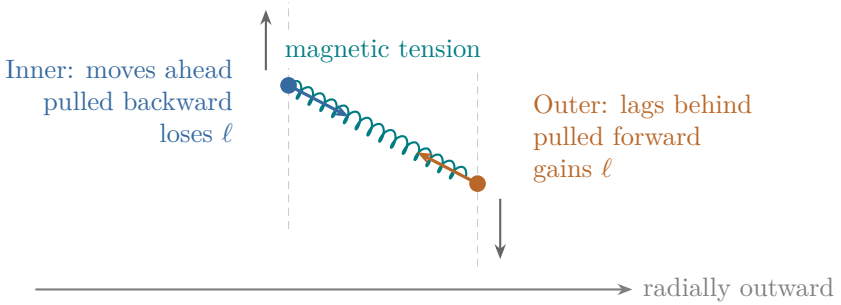


Figure 8: The spring analogy, viewed in a frame orbiting with the unperturbed elements. Gray arrows show relative orbital motion; colored arrows show tension. In the vertical-field MRI, the connected elements are at different heights, projected here onto the orbital plane.

Inner element: the backward pull removes angular momentum, allowing it to move farther inward and advance still farther ahead. **Outer element:** the forward pull adds angular momentum, allowing it to move farther outward and fall still farther behind. The separation and the magnetic torque reinforce one another: this is the MRI feedback.

The spring itself pulls the elements together. The instability comes from its *azimuthal torque*, which changes their orbits; an isolated element could not exchange angular momentum this way. The angular momentum lost by the inner element is gained by the outer one, giving the outward transport that accretion needs.

Magnetic tension supplies the coupling; differential rotation supplies the energy. A sufficiently weak magnetic spring can destabilize the orbital motion, while a sufficiently stiff spring suppresses the separation. We now construct a local model to calculate this competition.

From the disk to a shearing box

Choose a circular orbit of radius R_0 and follow it at angular velocity $\Omega_0 = \Omega_K(R_0)$. A small patch around this orbit becomes a **shearing box**: a local Cartesian domain rotating with the disk at its center.

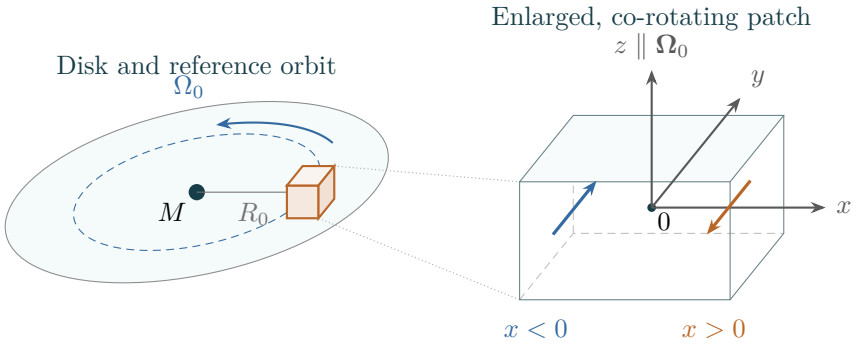


Figure 9: From a disk to a local box: x is radially outward, y is prograde (azimuthal), and z is vertical. Blue and orange arrows show the residual shear, not the full orbital velocity. Schematic following the supplied Squire notes, pp. 2–3.

With the box centered on $\phi = \Omega_0 t$ in the midplane, define

$$\boxed{x = R - R_0, \qquad y = R_0(\phi - \Omega_0 t), \qquad z = z_{\text{disk}}.} \tag{7.1}$$

The box is much smaller than R_0, so curvature can be neglected. These are local radial, azimuthal, and vertical directions, with $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}$. Write $\Omega \equiv \Omega_0$ below. Subtracting the rotation of the box leaves

$$U_{0y} \simeq R_0 \left.\frac{\mathrm{d}\Omega_K}{\mathrm{d}R}\right|_{R_0} x = -\frac{3}{2}\Omega x, \qquad \boldsymbol{U}_0 = -\frac{3}{2}\Omega x\, \hat{\boldsymbol{y}}. \tag{7.2}$$

Thus the inner side ($x < 0$) moves in $+y$ and the outer side ($x > 0$) in $-y$, relative to the box. We retain this shear, Coriolis forces, and the leading tidal force when making the local approximation.

Magnetic tension becomes the spring

Take constant density ρ_0, a uniform vertical field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$, and ideal MHD, ignoring vertical stratification. Horizontal displacements $\boldsymbol{\xi} = (\xi_x, \xi_y, 0) \propto e^{ik_z z}$ bend the field, linking elements at different heights. With no x or y dependence, these perturbations are incompressible.

Flux freezing gives the perturbed field:

$$\delta\boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}_0) = (\boldsymbol{B}_0 \cdot \boldsymbol{\nabla})\boldsymbol{\xi} = ik_z B_0 \boldsymbol{\xi}. \tag{7.3}$$

The resulting tension acceleration is

$$\frac{(\boldsymbol{B}_0 \cdot \boldsymbol{\nabla})\delta\boldsymbol{B}}{4\pi\rho_0} = -k_z^2 v_A^2 \boldsymbol{\xi} \equiv -K\boldsymbol{\xi}, \qquad v_A = \frac{B_0}{\sqrt{4\pi\rho_0}}. \tag{7.4}$$

Thus $K = k_z^2 v_A^2$ is the spring constant per unit mass. Here $\boldsymbol{B}_0 \cdot \delta\boldsymbol{B} = 0$ and there is no horizontal pressure gradient: magnetic tension is the horizontal restoring force to add to the orbital equations.

8 MRI derivation

Step 1: couple radial and azimuthal displacements

The local rotating-frame equations, with the force (7.4), are

$$\ddot{\xi}_x - 2\Omega\dot{\xi}_y = (3\Omega^2 - K)\xi_x, \tag{8.1}$$

$$\ddot{\xi}_y + 2\Omega\dot{\xi}_x = -K\xi_y. \tag{8.2}$$

The 2Ω terms are Coriolis terms; $3\Omega^2\xi_x$ is the Keplerian tidal acceleration from gravity plus centrifugal force; and $-K\boldsymbol{\xi}$ is magnetic tension. These are the local equations of orbital motion with a magnetic spring added.

As a check, set $K = 0$. Integrating (8.2) gives $\dot{\xi}_y + 2\Omega\xi_x = C$. Substitution into (8.1) yields $\ddot{\xi}_x + \Omega^2\xi_x = 2\Omega C$: a stable epicyclic oscillation about a shifted center. The positive tidal term alone does not imply an instability.

Step 2: seek an exponentially growing displacement

Put $\boldsymbol{\xi} \propto e^{st}$. Then

$$\begin{pmatrix} s^2 + K - 3\Omega^2 & -2\Omega s \\ 2\Omega s & s^2 + K \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = 0. \tag{8.3}$$

A nonzero displacement requires a zero determinant:

$$(s^2 + K - 3\Omega^2)(s^2 + K) + 4\Omega^2 s^2 = 0. \tag{8.4}$$

Expanding and collecting powers of s gives

$$\boxed{s^4 + (\Omega^2 + 2K)s^2 + K(K - 3\Omega^2) = 0.} \tag{8.5}$$

Step 3: read the instability condition

Treat (8.5) as a quadratic in $X = s^2$. Its discriminant is $\Omega^4 + 16\Omega^2 K > 0$, so both X roots are real. Their sum is negative. If their product $K(K - 3\Omega^2)$ is negative, one root is positive and gives an exponentially growing solution $s = +\sqrt{X}$.

Therefore the instability band is

$$\boxed{0 < K < 3\Omega^2 \quad \Longleftrightarrow \quad 0 < k_z v_A < \sqrt{3}\,\Omega,} \tag{8.6}$$

where we take $k_z > 0$ and v_A to be a positive speed. For $K > 3\Omega^2$, both X roots are negative: the modes oscillate. The endpoints of the band have zero exponential growth.

> Keplerian rotation is stable to the unmagnetized centrifugal instability, but unstable to suitable magnetically coupled perturbations. Tension enables the angular-momentum exchange; sufficiently strong tension suppresses the displacement.

9 MRI-driven stress, accretion, and the dynamo connection

The growing branch of (8.5) has

$$\gamma^2 = \frac{-(\Omega^2 + 2K) + \sqrt{\Omega^4 + 16\Omega^2 K}}{2}, \quad \gamma_{\max} = \frac{3}{4}\Omega, \quad (k_z v_A)_{\max} = \frac{\sqrt{15}}{4}\Omega. \tag{9.1}$$

Growth is fast, of order an orbital timescale. A real mode must fit in the disk and survive diffusion. Linear theory predicts initial growth, not the eventual turbulent stress.

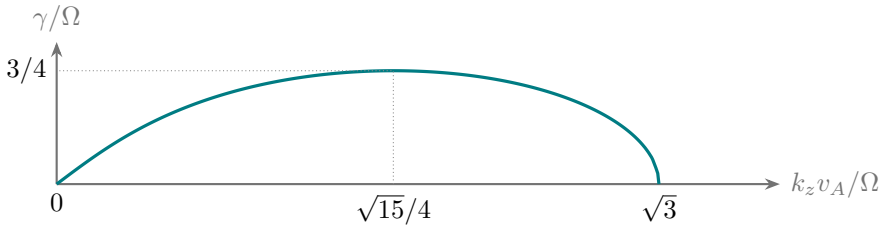


Figure 10: The Keplerian MRI growth rate. Weak coupling gives slow growth; sufficiently strong tension suppresses the instability.

The relevant diagnostic is a correlated stress

After separating mean mass advection, define the stress in the local constant-density model by

$$T_{R\phi} = \underbrace{\langle \rho u_R u_\phi \rangle}_{\text{Reynolds stress}} - \underbrace{\frac{\langle B_R B_\phi \rangle}{4\pi}}_{\text{magnetic correlation}}. \tag{9.2}$$

Here $\boldsymbol{u}$ is velocity relative to the mean orbital flow. Our convention is $T_{R\phi} > 0$ for outward angular-momentum flux $RT_{R\phi}$. The Maxwell stress includes the minus sign.

The same Ω effect discussed earlier winds radial field into an oppositely directed azimuthal component in Keplerian shear. The resulting $B_R B_\phi < 0$ contributes positive outward stress. Magnetic energy by itself does not tell us the sign or magnitude of this correlation. The rate of energy extraction from the shear is

$$\mathcal{P} = -T_{R\phi} R \frac{\mathrm{d}\Omega}{\mathrm{d}R} = \frac{3}{2}\Omega T_{R\phi}. \tag{9.3}$$

This energy feeds fluctuations and can later become heat or radiation. The global accretion rate also depends on the stress profile and boundaries.

Returning to field regeneration

MRI-driven motions act back on the field through induction. With imposed net flux, the system already has a background field. Without imposed net flux, sustained magnetic activity requires a successful regeneration process. That process need not reduce to a scalar $\alpha\Omega$ closure. The cyclic mean fields seen in stratified MRI simulations motivate measuring $\overline{\boldsymbol{B}}$ and $\boldsymbol{\mathcal{E}}$; a butterfly pattern alone does not identify the mechanism.

> The stress $T_{R\phi}$ controls angular-momentum transport and extraction of shear energy. The EMF $\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u}' \times \boldsymbol{b}'}$ controls induction of the mean field. Both are correlations, but they answer different questions.

10 Summary and recommended reading

What the three effects mean

Effect	Term in the mean equation	Physical role in the model
Ω	$\boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}})$	Differential rotation winds poloidal field into toroidal field.
α	$\boldsymbol{\nabla} \times (\alpha_{\mathrm{dyn}} \overline{\boldsymbol{B}})$	Correlated, statistically handed motions can couple and regenerate components.
β	$-\boldsymbol{\nabla} \times (\beta \boldsymbol{\nabla} \times \overline{\boldsymbol{B}})$	Positive constant β smooths gradients of the mean field.

The accretion parameter $\alpha_{\mathrm{SS}} \sim T_{R\phi}/P$ is dimensionless and is distinct from α_{dyn}, which has units of velocity. Turbulent diffusivity β is distinct from plasma beta, the pressure ratio. Large-scale dynamos need not all be scalar $\alpha\Omega$ dynamos.

Recommended reading

Dynamo physics and the classic physical pictures. F. Rincon, *Dynamo Theories* (2019): Sections 4.2–4.3.5 for the Ω and Parker pictures, the mean-field formalism, and basic dynamo models. Figures 26–28 are useful companions to the conceptual schematics here. Detailed closure calculations in Sections 4.3.6–4.3.8 are optional. Section 5.3 discusses MRI-related dynamos.

Primary accretion reading: the complete Armitage notes.

P. J. Armitage, *Lecture Notes on Accretion Disk Physics* (2022). The whole set remains strongly recommended. Begin with II.A–D for disk structure and MRI, then III.A–C for transport, disk evolution, and the accretion alpha prescription. Section IV develops the energetics.

MRI mechanism and the spring interpretation.

S. A. Balbus, *Magnetohydrodynamics of Protostellar Disks* (2009), Sections 4.1–4.2 and 4.4. For the foundational linear MRI paper cite Balbus & Hawley (1991).