P122 | Advanced Topics in Plasma Physics

Shocks and Particle Acceleration

Instructor: Muni Zhou

Shocks compress and heat plasma by converting bulk flow energy into thermal energy. Conservation of mass, momentum, and energy determines the changes across a shock, while plasma processes set its structure and thickness. Some particles can cross the shock repeatedly, gaining energy through scattering by magnetic fluctuations. The competition between energy gain and escape produces power-law spectra and helps explain the origin of cosmic rays.

Dartmouth College

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1 Hydro equations: a recap

The story in one line. Fluid that moves faster than sound piles up into a thin jump (a *shock*); conservation laws tell us what comes out of the jump (a hotter, denser, slower gas); and in space the jump is so thin and collisionless that it also launches a few particles to enormous energies (*cosmic rays*). We start from the equations that describe the gas on either side.

Notation. Each particle's velocity is $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$: $\boldsymbol{u} = \langle \boldsymbol{v} \rangle$ is the bulk (fluid) velocity and $\boldsymbol{w}$ is the thermal part, with $\langle \boldsymbol{w} \rangle = 0$. All fluid equations below are written for $\boldsymbol{u}$; the thermal motion enters only through the pressure, $P\,\delta_{ij} = \rho\,\langle w_i w_j \rangle$.

Assume isotropic pressure, no viscosity (ideal fluid), and no external forces.

$$\text{mass:} \qquad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{1.1}$$

$$\text{momentum (Euler):} \qquad \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{\boldsymbol{\nabla} P}{\rho}, \tag{1.2}$$

$$\text{energy:} \qquad \rho \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + P\,\boldsymbol{\nabla} \cdot \boldsymbol{u} = -\mathcal{L}. \tag{1.3}$$

Here ε is the internal energy per unit mass, $\mathrm{d}/\mathrm{d}t = \partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$ is the Lagrangian derivative (careful!), and $P\boldsymbol{\nabla} \cdot \boldsymbol{u}$ is the work done by compression or expansion. $\mathcal{L}$ is the heat-loss rate per unit volume (radiation, conduction); $\mathcal{L} = 0$ means *adiabatic*, which we assume from now on.

2 Conservative form

The equations above contain derivatives, so they break down exactly where we will need them: at a jump. The fix is to rewrite them as statements about *fluxes*, which remain meaningful even when ρ, u, P jump. Every conservation law can be written as

$$\frac{\partial}{\partial t}(Q) + \boldsymbol{\nabla} \cdot (\text{flux of } Q) = 0 \qquad \text{(or a source term, for external forces / heat gains or losses).}$$

- **Mass** is already in this form; $\rho\boldsymbol{u}$ is the mass flux.
- **Momentum:**

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j} R_{ij} = 0, \qquad R_{ij} = \rho\,\langle v_i v_j \rangle = \underbrace{\rho u_i u_j}_{\text{bulk (ram pressure)}} + \underbrace{\rho\,\langle w_i w_j \rangle}_{\text{thermal} = P\,\delta_{ij}}. \tag{2.1}$$

  Here we split each particle's velocity as $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ and averaged over particles: the cross terms vanish because $\langle \boldsymbol{w} \rangle = 0$, and isotropy gives $\rho\,\langle w_i w_j \rangle = P\,\delta_{ij}$. R_{ij}, the Reynolds stress tensor, is the flux of momentum in direction i across a surface whose outward normal is j.

- **Energy:**

$$\frac{\partial}{\partial t}\Big[\rho\Big(\frac{u^2}{2} + \varepsilon\Big)\Big] + \boldsymbol{\nabla} \cdot \Big[\rho \boldsymbol{u}\Big(\frac{u^2}{2} + h\Big)\Big] = 0, \qquad h = \varepsilon + \frac{P}{\rho}. \tag{2.2}$$

  A fluid element of mass $\mathrm{d}m$ carries internal energy $\varepsilon\,\mathrm{d}m$ and pushes neighboring gas aside, doing pressure work $P\,\mathrm{d}V = (P/\rho)\,\mathrm{d}m$. Together these give $h\,\mathrm{d}m$, so the energy flux contains the enthalpy $h = \varepsilon + P/\rho$. The stored energy density still contains ε; the extra P/ρ accounts for pressure work during transport.

Why bother? The conservative form is useful numerically, and it is exactly what we need for shocks (Sec. 4): it still makes sense when the fluid quantities are discontinuous.

3 Wave steepening into shocks

Before computing what a shock does, we ask why shocks form at all. The answer: a big enough sound wave cannot stay smooth.

Linearizing the hydro equations gives sound waves with adiabatic sound speed

$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_s$$

(adiabatic if the evolution is fast compared with non-adiabatic processes). In the linear calculation we set $\boldsymbol{u}_0 = 0$, kept $\boldsymbol{u}_1 \neq 0$ small, and *neglected* the second-order term $(\boldsymbol{u}_1 \cdot \boldsymbol{\nabla})\boldsymbol{u}_1$.

What if u_1 is not small? Take the 1D momentum equation and compare terms:

$$\frac{\partial u}{\partial t} + \underbrace{u\frac{\partial u}{\partial x}}_{\sim u^2/L} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\sim c_s^2/L}.$$

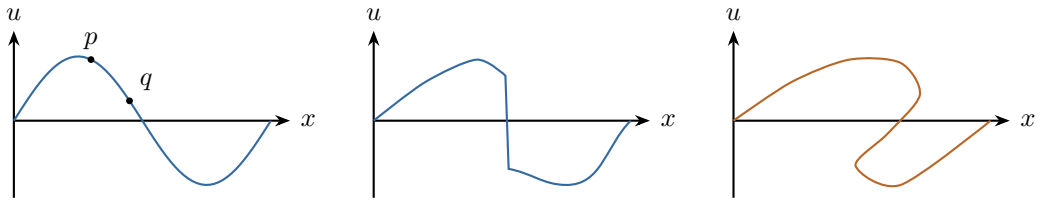
If $u^2 \gg c_s^2$, the pressure term can be dropped:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{3.1}$$

Now follow the curve $\mathrm{d}x/\mathrm{d}t = u$ in the (x, t) plane. Along it,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x} = 0.$$

This is the *method of characteristics*: the PDE becomes an ODE on special curves. Here the characteristic is simply the Lagrangian track of a fluid element, so *each fluid element keeps its velocity.*



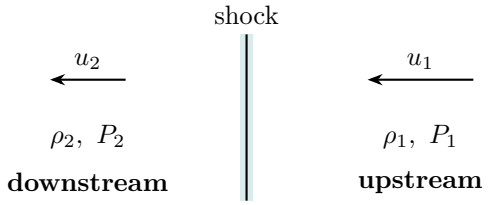
$t = 0$ later: steepens multi-valued!

Since $u(p) > u(q)$, point p catches up with q and the profile steepens. Eventually it would overturn and become multi-valued, which is unphysical. **Bottom line:** $u\,\partial u/\partial x$ steepens the wave until the neglected pressure term kicks in, and a shock forms.

4 Shock jump conditions

So any non-infinitesimal perturbation with $u > c_s$ develops a shock. We do not need to resolve the messy interior of the jump: whatever happens inside, mass, momentum, and energy going in must come out. That is enough to predict the downstream state.

- **Math:** a discontinuity of hydro quantities.
- **Physics:** a jump on scales $\ll$ the size of the system.



Work in the shock frame, where the flow is steady. Then $\partial Q/\partial t = 0$ in $\partial_t Q + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$, so $\partial J/\partial x = 0$ and the flux is the same on both sides: $J_2 = J_1$. Writing $[X] \equiv X_2 - X_1$:

$$\text{mass:} \qquad [\rho u_x] = 0 \quad (\text{i.e. } \rho_1 u_{1x} = \rho_2 u_{2x}), \tag{4.1}$$

$$\text{momentum:} \qquad \left[\rho u_x^2 + P\right] = 0, \qquad [\rho u_x u_y] = [\rho u_x u_z] = 0, \tag{4.2}$$

$$\text{energy:} \qquad \left[\rho u_x\left(\tfrac{1}{2}u^2 + h\right)\right] = 0. \tag{4.3}$$

There are two options.

(A) Zero mass flux, $u_{1x} = u_{2x} = 0$: tangential discontinuity. No fluid crosses the surface. Momentum gives $P_1 = P_2$, while the tangential velocities are arbitrary (if they are also equal, it is a *contact discontinuity*).

(B) Non-zero mass flux: a shock. Dividing by ρu_x,

$$[\rho u_x] = 0, \quad \left[\tfrac{1}{2}u^2 + h\right] = 0, \quad [u_y] = [u_z] = 0, \quad \left[P + \rho u_x^2\right] = 0.$$

The tangential velocity is continuous, so we can move to a frame where it vanishes and $u = u_x$.

4.1 Solving for the jump

Count: three equations for three unknowns ρ_2, u_2, P_2, using $h = \varepsilon + P/\rho = \gamma P/[(\gamma-1)\rho]$ with γ the adiabatic index. Define the upstream Mach number

$$M_1 = \frac{u_1}{c_{s,1}} = \frac{u_1}{\sqrt{\gamma P_1/\rho_1}} \qquad (>1 \text{ for shocks}).$$

Solving gives the **jump conditions**:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma+1}{2/M_1^2 + (\gamma-1)} > 1, \qquad \frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}. \tag{4.4}$$

For a strong shock ($M_1 \to \infty$):

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}\ (= 4 \text{ for } \gamma = 5/3), \qquad k_B T_2 \to \frac{2(\gamma-1)}{(\gamma+1)^2}\, m u_1^2 \ \left(= \tfrac{3}{16} m u_1^2\right).$$

The compression saturates at a factor 4, but the temperature keeps growing as u_1^2: the bulk kinetic energy is **transformed into heat**.

4.2 Fun facts

1. $M_1 > 1$ but $M_2 = u_2/c_{s,2} < 1$: supersonic in, subsonic out (supersonic planes!).

Downstream Mach number. Using $r = \rho_2/\rho_1 = u_1/u_2$ and $c_s^2 = \gamma P/\rho$,

$$\frac{M_2}{M_1} = \sqrt{\frac{P_1}{rP_2}} \quad \Rightarrow \quad \boxed{M_2 = \sqrt{\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}} < 1}.$$

For $M_1 > 1$, the denominator inside the square root exceeds the numerator by $(\gamma+1)(M_1^2-1) > 0$.

2. The entropy per particle increases, $s_2 > s_1$, if and only if $M_1 > 1$. For an ideal gas,

$$\frac{s_2 - s_1}{k_B} = \frac{1}{\gamma-1}\ln\!\left[\frac{P_2}{P_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}\right].$$

Substituting the jump conditions and differentiating gives

$$\frac{\mathrm{d}}{\mathrm{d}(M_1^2)}\left(\frac{s_2-s_1}{k_B}\right) = \frac{2\gamma(M_1^2-1)^2}{M_1^2\left[(\gamma-1)M_1^2+2\right]\left[2\gamma M_1^2-(\gamma-1)\right]} > 0 \quad (M_1 > 1).$$

At $M_1 = 1$ there is no jump and $s_2 - s_1 = 0$; hence $M_1 > 1$ gives $s_2 > s_1$.

3. If we took $M_1 < 1$ and applied the same jump conditions, we would get $u_1 < u_2$ and $T_2 < T_1$: heat would be turned into ordered energy/work. This violates the 2nd law of thermodynamics, so there are no "expansion shocks."

5 MHD shocks

Astrophysical gases are plasmas threaded by magnetic fields, so we repeat the same recipe (conservative form → equal fluxes) with $\boldsymbol{B}$ included. Nothing conceptually new happens: the field just adds its own pressure, tension, and energy flux. The MHD equations can be written in conservative form; momentum, for example, becomes

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\Big[P\delta_{ij} + \rho u_i u_j - \frac{1}{4\pi}\Big(B_iB_j - \frac{B^2}{2}\delta_{ij}\Big)\Big] = 0, \tag{5.1}$$

where the last term is the Maxwell stress tensor.

MHD shocks must be both supersonic and super-Alfvénic, $v_A = B/\sqrt{4\pi\rho}$ (i.e. super-fast-magnetosonic). In most astrophysical cases $\beta = P_{\rm gas}/P_{\rm mag} \sim c_s^2/v_A^2 \gtrsim 1$, so supersonic is often enough.

Jump conditions (n = normal, t = tangential to the shock). Besides mass, momentum, and energy, Maxwell's equations require

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \;\Rightarrow\; [B_n] = 0, \qquad -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{E} \;\Rightarrow\; [E_t] = 0.$$

With ideal MHD, $\boldsymbol{E} = -\boldsymbol{u}\times\boldsymbol{B}/c$, the second becomes $[B_n u_t - B_t u_n] = 0$. In total:

$$[\rho u_n] = 0 \qquad \text{mass} \tag{5.2}$$

$$\Big[\rho u_n u_t - \tfrac{1}{4\pi}B_nB_t\Big] = 0 \qquad \text{momentum } \perp \text{ shock normal} \tag{5.3}$$

$$\Big[P + \rho u_n^2 + \tfrac{1}{8\pi}(B_t^2 - B_n^2)\Big] = 0 \qquad \text{momentum } \parallel \text{ shock normal} \tag{5.4}$$

$$\Big[\rho u_n\big(\tfrac{u^2}{2}+h\big) + \tfrac{1}{4\pi}\big(u_nB^2 - B_n(\boldsymbol{u}\cdot\boldsymbol{B})\big)\Big] = 0 \qquad \text{energy} \tag{5.5}$$

$$[B_n u_t - B_t u_n] = 0, \qquad [B_n] = 0 \qquad \text{induction, } \boldsymbol{\nabla}\cdot\boldsymbol{B} \tag{5.6}$$

5.1 Two simple cases

Choose a frame with $u_t = 0$.

(A) Parallel shock, $B_{t,1} = 0$. Then $B_{t,2} = 0$ as well. B_n decouples and we are back to the hydro equations: the field is along the flow and does not care about compression.

(B) Perpendicular shock, $B_{n,1} = 0$. The induction condition gives flux freezing,

$$\frac{B_{t2}}{B_{t1}} = \frac{\rho_2}{\rho_1} = \frac{u_{1n}}{u_{2n}},$$

and then

$$[\rho u] = 0, \qquad \left[P + \rho u^2 + \frac{B_t^2}{8\pi}\right] = 0, \qquad \left[\frac{1}{2}u^2 + h + \frac{B_t^2}{4\pi\rho}\right] = 0.$$

So the B field simply enters as an extra pressure and energy density.

6 Shock thickness and collisionless shocks

So far the shock was an infinitely thin line. What sets its real thickness? Answering this reveals something surprising about shocks in space, and it is the bridge to particle acceleration.

Pressure becomes important as compression heats the gas. Across a layer of thickness Δ,

$$\text{inertia} \sim \frac{u\,\Delta u}{\Delta}, \qquad \text{pressure} \sim \frac{\Delta P}{\rho\,\Delta}, \qquad \text{viscosity} \sim \frac{\nu\,\Delta u}{\Delta^2},$$

where ν is the kinematic viscosity. Viscosity grows faster as the layer narrows. Balancing it against steepening gives

$$\frac{u\,\Delta u}{\Delta} \sim \frac{\nu\,\Delta u}{\Delta^2} \quad \Rightarrow \quad \boxed{\Delta \sim \frac{\nu}{u}}.$$

If collisions provide this viscosity, the thickness is of order the mean free path:

$$\Delta \sim \lambda_{\rm mfp} = \frac{1}{n\sigma}.$$

Earth's atmosphere. $n \sim 3 \times 10^{19}\,\mathrm{cm}^{-3}$ and $\sigma \sim$ a few πa_B^2 (Bohr radius), giving $\lambda_{\rm mfp} \sim 10^{-5}\,\mathrm{cm}$. Tiny: a sound-wave shock is a true discontinuity.

Supernova shock. The medium is ionized, $n \sim 1\,\mathrm{cm}^{-3}$, and for the cross section take the size of a nucleus, $\sigma \sim \pi R_N^2 \sim \pi(2 \times 10^{-13}\,\mathrm{cm})^2$. Then $\lambda_{\rm mfp} \sim 10^{25}\,\mathrm{cm}$, much larger than the SN remnant radius ($\sim 1\,\mathrm{pc} \sim 3 \times 10^{18}\,\mathrm{cm}$).

SN shocks are **not** mediated by binary collisions, but by electromagnetic fields: they are **collisionless shocks**. Their thickness is set by the ion Larmor radius,

$$\Delta \sim r_L \sim \frac{m_p v c}{eB} \sim 10^{11}\,\mathrm{cm}\left(\frac{v}{10^4\,\mathrm{km/s}}\right)\left(\frac{10^{-6}\,\mathrm{G}}{B}\right).$$

Fun facts about collisionless shocks.

1. $T_e \neq T_i$ downstream: the electron–ion equilibration time is much longer than the system evolution time.
2. Non-thermal particles: even within one species, the equilibration time is longer than the system evolution, so the distribution need not be Maxwellian. It shows a thermal peak plus a *non-thermal tail*, often a power law $\mathrm{d}N/\mathrm{d}E \propto E^{-s}$ with s the power-law slope.

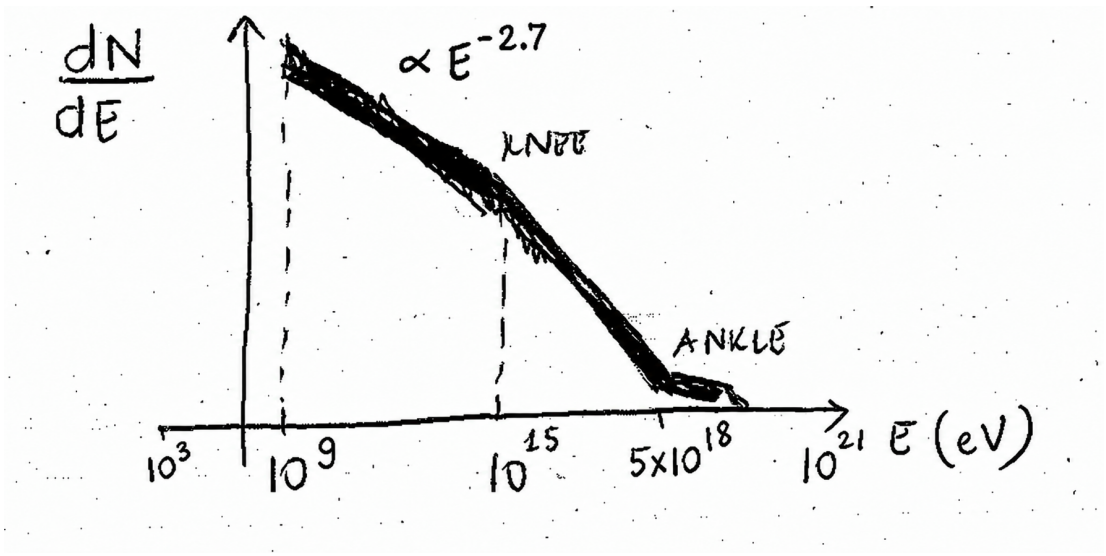
7 Cosmic rays

That non-thermal tail is not a curiosity: we observe it directly, arriving at Earth as cosmic rays. The rest of the lecture asks whether shocks can make them.

Cosmic rays (CRs) are charged particles originating beyond Earth (mostly hadrons), with energies up to $\sim 10^{20}$–10^{21} eV ($1\,\mathrm{eV} \approx 1.6 \times 10^{-12}\,\mathrm{erg} = 1.6 \times 10^{-19}\,\mathrm{J}$).

Compare with a tennis ball: $m \sim 0.1\,\mathrm{kg}$, $v \sim 100\,\mathrm{km/hr} \sim 30\,\mathrm{m/s}$, so $E \sim \frac{1}{2}(0.1)(30)^2 \sim 50\,\mathrm{J} \lesssim 10^{21}\,\mathrm{eV}$. A single proton carries the energy of a tennis serve!

Above GeV energies, the spectrum is approximately $\mathrm{d}N/\mathrm{d}E \propto E^{-2.7}$ up to the "knee" at $E \sim 3 \times 10^{15}\,\mathrm{eV}$. It steepens above the knee and becomes flatter at the "ankle" near $5 \times 10^{18}\,\mathrm{eV}$. We think CRs up to the knee are produced at **supernova-remnant shocks**.



Schematic cosmic-ray spectrum on logarithmic axes; normalization arbitrary and low-energy shape omitted.

Two questions:

1. Can SN shocks accelerate and confine PeV particles?
2. What is the spectrum $\mathrm{d}N/\mathrm{d}E$ of the accelerated cosmic rays?

7.1 Question 1: the Hillas criterion

To be accelerated, a particle must stay in the system: $r_L < L$. For 3 PeV protons in $B \sim \mu$G,

$$r_L \sim \frac{E}{eB} \sim \frac{(3 \times 10^{15})(1.6 \times 10^{-12})\,\mathrm{erg}}{(4.8 \times 10^{-10})(10^{-6})} \sim 10^{19}\,\mathrm{cm}, \qquad L \sim 1\,\mathrm{pc} \sim 3 \times 10^{18}\,\mathrm{cm}.$$

This seems insufficient, even when we include the 4× compression of the ISM field at the shock. **Solution? Plasma physics!** (Bell 2004, 2005): CRs accelerated at the shock drive a plasma instability that can amplify the upstream field by up to $\sim 30\times$, reducing r_L well below L.

8 Particle acceleration: Fermi's basic idea (Fermi II)

Question 1 asked whether a shock can *hold* a particle long enough. Question 2 asks *how* the particle gains energy, and what spectrum results. We build up in two steps: first Fermi's original, somewhat inefficient mechanism, then the version a shock provides. Enrico Fermi's idea:

- Magnetized clouds move at $V \ll c$.
- Particles bounce off clouds as off rigid walls: the reflection is elastic in the cloud frame.

A particle has energy E and momentum p at angle θ to $\hat{x}$; the cloud moves along $-\hat{x}$ with speed V (head-on when $\cos\theta > 0$). Primed quantities are in the cloud frame:

$$E' = \gamma_V(E + Vp_x), \qquad p'_x = \gamma_V\Big(p_x + \frac{VE}{c^2}\Big), \qquad \gamma_V = \frac{1}{\sqrt{1 - V^2/c^2}}, \qquad p_x = p\cos\theta.$$

In the cloud frame the collision only flips p'_x: $E'_1 = E'$, $p'_{x1} = -p'_x$. Transforming back to the lab, for relativistic particles ($E = pc$) and to 2nd order in V/c,

$$\frac{\Delta E}{E} \equiv \frac{E_1 - E}{E} \simeq \frac{2V\cos\theta}{c} + \frac{2V^2}{c^2}. \tag{8.1}$$

The first term can have either sign. Over many collisions we average over θ. The average is *non-zero* because head-on collisions are more frequent (you run into more cars driving against you):

$$f(\theta)\,\mathrm{d}\theta = \frac{1}{2}\Big(1 + \frac{V\cos\theta}{c}\Big)\sin\theta\,\mathrm{d}\theta,$$

where the bracket is the flux factor (relative velocity). Then

$$\frac{\langle\Delta E\rangle}{E} = \frac{8}{3}\frac{V^2}{c^2} \qquad \text{(Fermi II: second order in } V/c\text{, so kind of slow).}$$

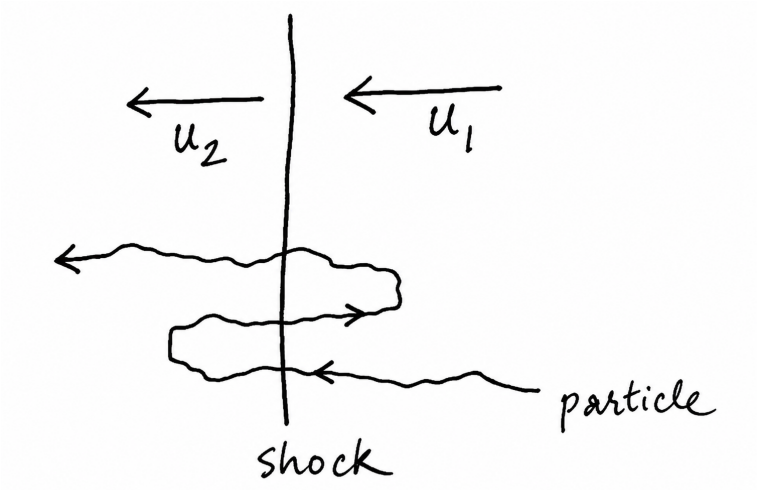
Astro application: turbulence-driven acceleration, where turbulent eddies play the role of clouds.

Question: the particle energy is increasing; where does the energy come from? (From the bulk kinetic energy of the clouds, which recoil slightly on each bounce.)

9 Shock acceleration (Fermi I)

Fermi II is slow because gains and losses nearly cancel: the net effect survives only at second order. A shock removes the cancellation. Seen from either side, the fluid on the other side is always approaching. At a shock, upstream and downstream fluids move towards each other, so they act as *always converging* clouds. With compression ratio r,

$$\frac{V_d}{V_u} = \frac{1}{r} = \frac{\rho_u}{\rho_d}, \qquad V \equiv V_u - V_d = \frac{r-1}{r}V_u \quad \text{(relative speed).}$$



Downstream (left), upstream (right); arrows show repeated shock crossings.

If particles get efficiently isotropized (scattered by magnetic fluctuations) in one region, they meet the other region's flow always head-on, so the energy gain is *systematic*. Contrast Fermi II, where tail-on collisions lead to energy loss.

Probability of crossing. For isotropic particles, the flux of those crossing the shock at angle θ is

$$f(\theta)\,\mathrm{d}\theta = 2\cos\theta\,\sin\theta\,\mathrm{d}\theta, \qquad \cos\theta \geq 0,$$

($\cos\theta$ is the flux factor, $\sin\theta$ the isotropic weight). Each crossing gives $\Delta E/E \simeq V\cos\theta/c$; averaging and doing a full cycle D→U→D,

$$\left.\frac{\langle\Delta E\rangle}{E}\right|_{D\to U\to D} = \frac{4}{3}\frac{V}{c} \qquad \textbf{first order!}$$

9.1 How does this lead to a power law $\mathrm{d}N/\mathrm{d}E \propto E^{-s}$?

Each cycle multiplies a particle's energy by a fixed factor, but at each cycle some particles are lost. Particles that survive many cycles are rare but very energetic, and this trade-off is exactly what produces a power law. After k acceleration cycles:

(a) $E = A^k E_0$, with $A = 1 + \frac{4}{3}\frac{V}{c}$.

(b) If P is the probability to remain in the acceleration region per cycle, $N = N_0 P^k$ ($< N_0$).

Eliminating $k = \ln(E/E_0)/\ln A$:

$$N(\geq E) = N_0 P^{\ln(E/E_0)/\ln A} = N_0\Big(\frac{E}{E_0}\Big)^{\ln P/\ln A} \quad \text{(cumulative)},$$

$$\Rightarrow \quad \frac{\mathrm{d}N}{\mathrm{d}E} \propto E^{-s}, \qquad s = 1 - \frac{\ln P}{\ln A} \quad \text{(differential)}. \tag{9.1}$$

A power law arises from the competition between a fixed fractional gain and a fixed escape probability.

9.2 How to compute P?

Particles can only escape downstream, where V_d carries them away from the shock. In steady state the fluxes balance:

$$J_+ = J_- + J_\infty,$$

with J_+ entering D, J_- returning to U, and J_∞ escaping to downstream infinity. The retention probability is $P = J_-/J_+ = J_-/(J_- + J_\infty)$, with

$$J_- = \int_{\cos\theta>0} \frac{\mathrm{d}\Omega}{4\pi}\, nc\cos\theta = \frac{nc}{4}, \qquad J_\infty = nV_d \quad \text{(carried away at } V_d\text{)}.$$

So $P \simeq 1 - 4V_d/c$, and

$$s = 1 - \frac{\ln P}{\ln A} \simeq 1 + \frac{3V_d}{V} = 1 + \frac{3}{r-1}.$$

Strong shocks have $r = 4 \;\Rightarrow\; s = 2$: **universal!**

Fermi I is fast, and leads to power-law particles with $s = 2$, independent of the details of the scattering. (The observed $E^{-2.7}$ is steeper, partly because of energy-dependent escape from the Galaxy.)

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